1. i. By sketching the curves $y = \ln x$ and $y = 8 - 2x^2$ on a single diagram, show that the equation

$$\ln x = 8 - 2x^2$$

has exactly one real root.

[3]

ii. Explain how your diagram shows that the root is between 1 and 2.

[1]

iii. Use the iterative formula

$$x_{n+1} = \sqrt{4 - \tfrac{1}{2}\ln x_n},$$

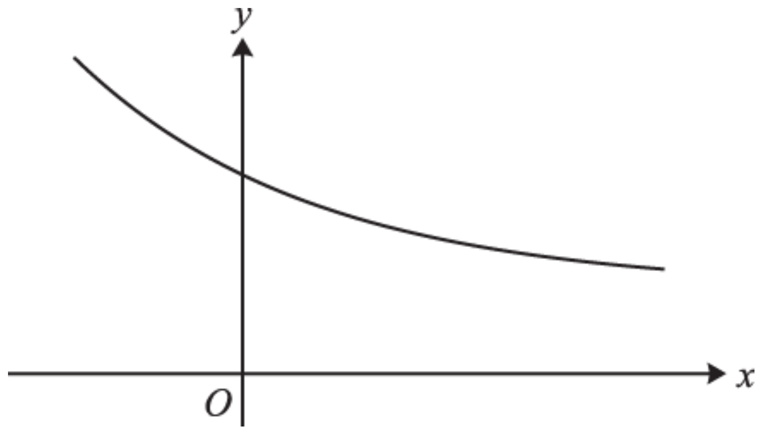
with a suitable starting value, to find the root. Show all your working and give the root correct to 3 decimal places.

[4]

iv. The curves $y = \ln x$ and $y = 8 - 2x^2$ are each translated by 2 units in the positive x-direction and then stretched by scale factor 4 in the y-direction. Find the coordinates of the point where the new curves intersect, giving each coordinate correct to 2 decimal places.

[3]

2.



The diagram shows the curve $y = f(x)$, where f is the function defined for all real values of x by

$$f(x) = 3 + 4e^{-x}.$$

i. State the range of f.

[1]

ii. Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}.

[4]

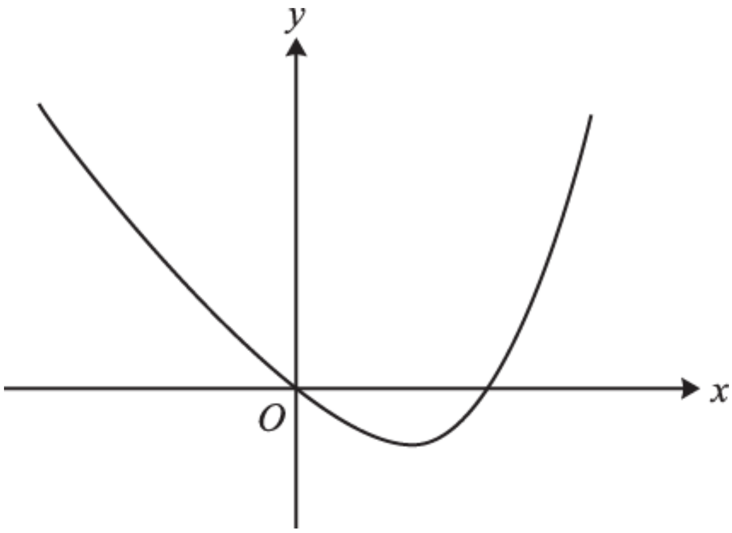
iii. The straight line $y = x$ meets the curve $y = f(x)$ at the point P. By using an iterative process based on the equation $x = f(x)$, with a starting value of 3, find the coordinates of the point P. Show all your working and give each coordinate correct to 3 decimal places.

[4]

iv. How is the point P related to the curves $y = f(x)$ and $y = f^{-1}(x)$?

[1]

3.



The diagram shows the curve $y = x^4 - 8x$.

i. By sketching a second curve on the copy of the diagram, show that the equation

$$x^4 + x^2 - 8x - 9 = 0$$

has two real roots. State the equation of the second curve.

[2]

ii. The larger root of the equation $x^4 + x^2 - 8x - 9 = 0$ is denoted by α.

a. Show by calculation that $2.1 < \alpha < 2.2$.

[2]

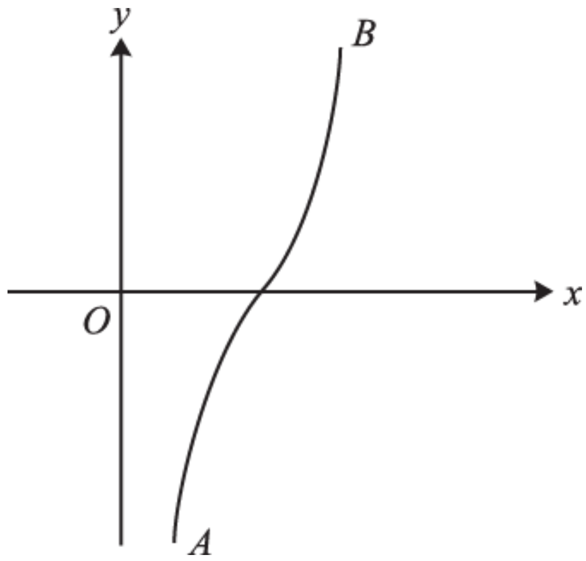
b. Use an iterative process based on the equation

$$x = \sqrt[4]{9 + 8x - x^2},$$

with a suitable starting value, to find *a* correct to 3 decimal places. Give the result of each step of the iterative process.

[4]

4.



The diagram shows the curve $y = 8\sin^{-1}\left(x - \frac{3}{2}\right)$. The end-points *A* and *B* of the curve have coordinates (*a*, −4π) and (*b*, 4π) respectively.

i. State the values of *a* and *b*.

[2]

ii. It is required to find the root of the equation $8\sin^{-1}\left(x - \frac{3}{2}\right) = x$.

a. Show by calculation that the root lies between 1.7 and 1.8.

[3]

b. In order to find the root, the iterative formula

$$x_{n+1} = P + \sin(qx_n),$$

with a suitable starting value, is to be used. Determine the values of the constants *p* and *q* and hence find the root correct to 4 significant figures. Show the result of each step of the iteration process.

[5]

5. i. By sketching the curves $y = x(2x + 5)$ and $y = \cos^{-1}x$ (where y is in radians) in a single diagram, show that the equation $x(2x + 5) = \cos^{-1}x$ has exactly one real root.

[3]

ii. Use the iterative formula

$$x_{n+1} = \frac{\cos^{-1}x_n}{2x_n + 5} \text{ with } x_1 = 0.25$$

to find the root correct to 3 significant figures. Show the result of each iteration correct to at least 4 significant figures.

[4]

iii. Two new curves are obtained by transforming each of the curves $y = x(2x + 5)$ and $y = \cos^{-1}x$ by the pair of transformations:

reflection in the x-axis followed by reflection in the y-axis.

State an equation of each of the new curves and determine the coordinates of their point of intersection, giving each coordinate correct to 3 significant figures.

[4]

6. The equation $x^3 - x^2 - 5x + 10 = 0$ has exactly one real root α.

(a) Show that the Newton-Raphson iterative formula for finding this root can be written as

$$x_{n+1} = \frac{2x_n^{\ 3} - x_n^{\ 2} - 10}{3x_n^{\ 2} - 2x_n - 5}.$$

[3]

(b) Apply the iterative formula in part **(a)** with initial value $x_1 = -3$ to find x_2, x_3, x_4 correct to 4 significant figures. [1]

(c) Use a change of sign method to show that $\alpha = -2.533$ is correct to 4 significant figures. [3]

(d) Explain why the Newton-Raphson method with initial value $x_1 = -1$ would not converge to α. [2]

7.

(a) By sketching the graphs of $y = \frac{5}{x^2}$ and $y = |2 - 4x|$ on a single diagram, show that the equation

$$\frac{5}{x^2} = |2 - 4x| \qquad \text{(A)}$$

has exactly two real roots. [3]

(b) Show that the positive root α of equation (A) satisfies the equation $f(x) = 0$, where [1]

$$f(x) = 4x^3 - 2x^2 - 5 .$$

(c) Hence show that a Newton-Raphson iterative formula for finding α can be written in the form [3]

$$x_{n+1} = \frac{8x_n^3 - 2x_n^2 + 5}{12x_n^2 - 4x_n}.$$

(d) Use this iterative formula, with initial value $x_1 = 1$, to find the value of α correct to 3 decimal places. Show the result of each iteration. [3]

A student claims that the iterative formula from part (c) can be used to find the negative root of equation (A) provided that a suitable initial value is chosen.

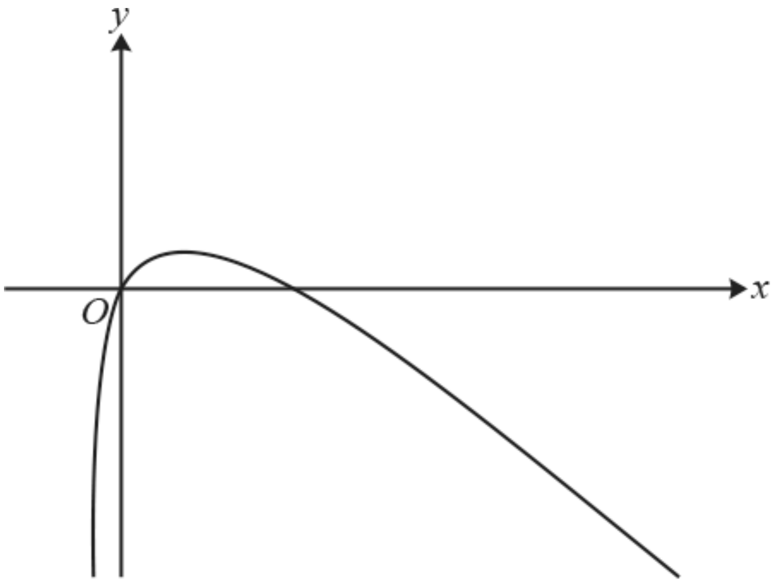
(e) Explain why the student's claim is incorrect. [1]

8. The equation $x^3 - x - 2 = 0$ has exactly one real root α.

(a) Use the iterative formula $x_{n+1} = \sqrt[3]{x_n + 2}$ with $x_1 = 1$ to find α correct to 4 significant figures, showing the result of each iteration. [3]

(b) An alternative iterative formula is $x_{n+1} = F(x_n)$, where $F(x_n) = \frac{x_n + 2}{x_n^2}$. By considering $F'(\alpha)$, explain why this iterative process will not converge to α. [3]

9.



The diagram shows the graph of $f(x) = \ln(3x + 1) - x$, which has a stationary point at $x = \alpha$. A student wishes to find the non-zero root β of the equation $\ln(3x + 1) - x = 0$ using the Newton-Raphson method.

(a) **(i)** Determine the value of α. **[3]**

(ii) Explain why the Newton-Raphson method will fail if α is used as the initial value. **[1]**

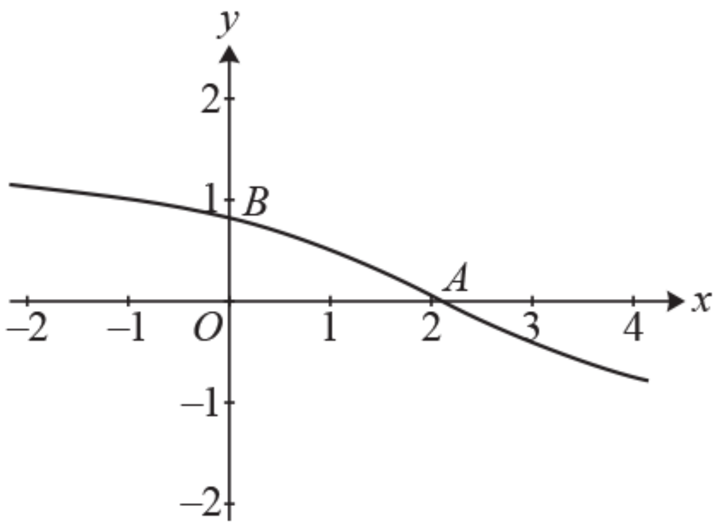
(b) Show that the Newton-Raphson iterative formula for finding β can be written as

$$x_{n+1} = \frac{3x_n - (3x_n + 1)\ln(3x_n + 1)}{2 - 3x_n}.$$ **[3]**

(c) Apply the iterative formula in part **(b)** with initial value $x_1 = 1$ to find the value of β correct to 5 significant figures. You should show the result of each iteration. **[3]**

(d) Use a change of sign method to verify that the value of β found in part **(c)** is correct to 5 significant figures. **[3]**

10.



The diagram shows the graph of $y = -\tan^{-1}\left(\frac{1}{2}x - \frac{1}{3}\pi\right)$, which crosses the x-axis at the point A and the y-axis at the point B.

(a) Determine the coordinates of the points A and B. [3]

(b) Give full details of a sequence of three geometrical transformations which transform the graph of

$y = \tan^{-1}x$ to the graph of $y = -\tan^{-1}\left(\frac{1}{2}x - \frac{1}{3}\pi\right)$. [3]

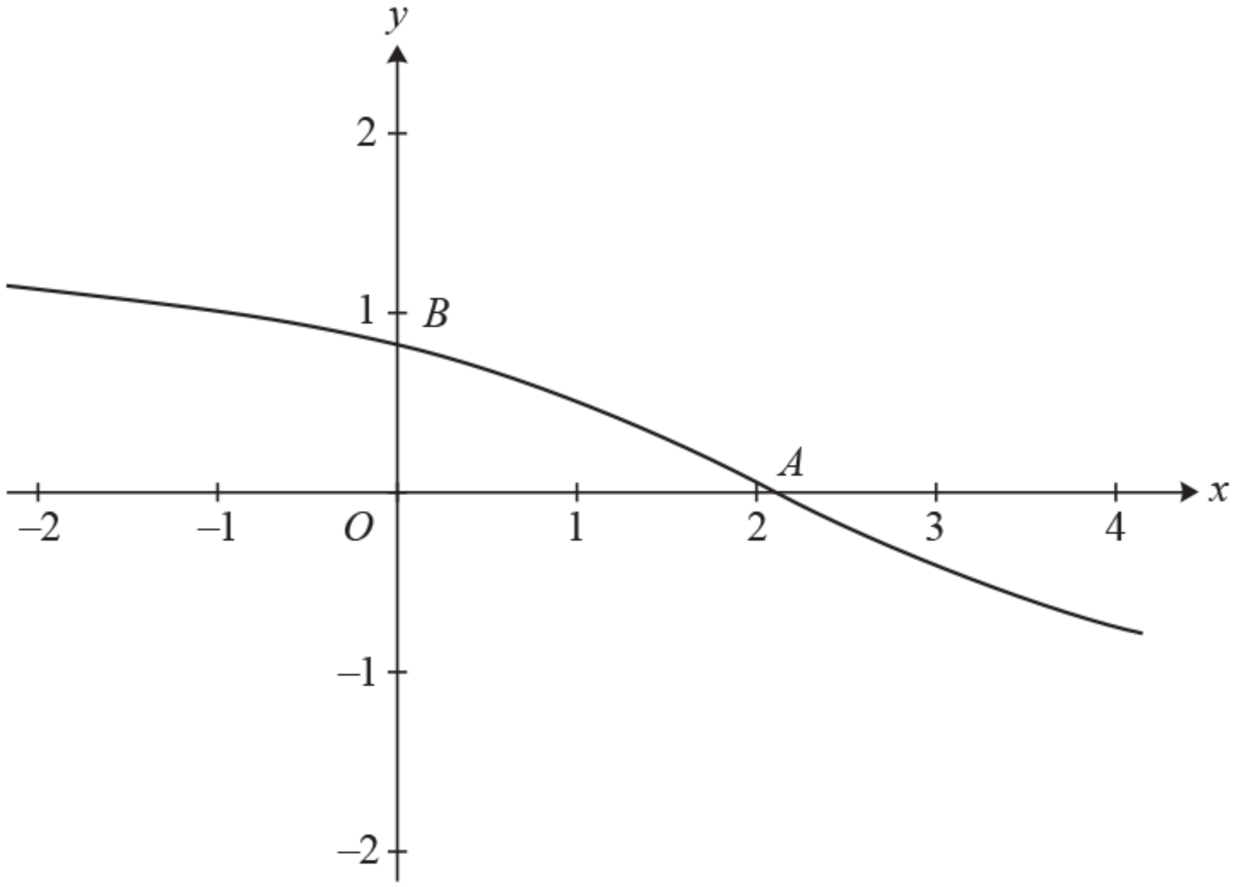
The equation $x = -\tan^{-1}\left(\frac{1}{2}x - \frac{1}{3}\pi\right)$ has only one root.

(c) Show by calculation that this root lies between $x = 0$ and $x = 1$. [2]

(d) Use the iterative formula $x_{n+1} = -\tan^{-1}\left(\frac{1}{2}x_n - \frac{1}{3}\pi\right)$, with a suitable starting value, to find the root correct to 3 significant figures. Show the result of each iteration. [3]

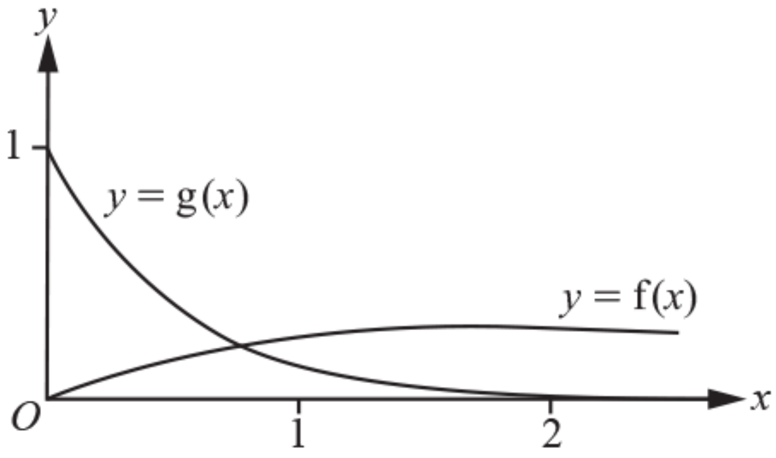
Using the diagram below, show how the iterative process converges to the root.

(e)



[2]

11.



The functions f(x) and g(x) are defined for $x \geq 0$ by $f(x) = \dfrac{x}{x^2+3}$ and $g(x) = e^{-2x}$. The diagram shows the curves $y = f(x)$ and $y = g(x)$. The equation $f(x) = g(x)$ has exactly one real root α.

(a) Show that α satisfies the equation $h(x) = 0$, where $h(x) = x^2 + 3 - xe^{2x}$. [2]

(b) Hence show that a Newton-Raphson iterative formula for finding α can be written in the form

$$x_{n+1} = \frac{x_n^2(1-2e^{2x_n})-3}{2x_n-(1+2x_n)e^{2x_n}}.$$ [5]

(c) Use this iterative formula, with a suitable initial value, to find α correct to 3 decimal places. Show the result of each iteration. [3]

(d) **In this question you must show detailed reasoning.**

Find the exact value of x for which $fg(x) = \frac{2}{13}$. [6]

END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Part marks and guidance	
1		i	Sketch more or less correct $y = \ln x$	B1	existing for positive and negative y; no need to indicate (1, 0); ignore any scales given on axes; condone graph touching y-axis but B0 if it crosses y-axis	
		i	Sketch more or less correct $y = 8 - 2x^2$	B1	(roughly) symmetrical about y-axis; extending, if minimally, into quadrants for which $y < 0$; no need to indicate (± 2, 0), (0, 8); assess each curve separately	
		i	Indicate intersection by some mark on diagram (just a 'blob' sufficient) of by statement in words away from diagram	B1	needs each curve to be (more or less) correct in the first quadrant and on curves being related to each other correctly there <u>Examiner's Comments</u> Examiners were reasonably tolerant in assessing the two sketches but, even so, many attempts were not as assured as they should have been. Many attempts at the sketch of $y = 8 - 2x^2$ were far from being symmetrical about the y-axis and a few were even straight lines. Many attempts at the sketch of $y = \ln x$ clearly touched the negative y-axis; others passed through the origin, existed for negative values of x or had the wrong curvature. For full credit, the parabola had to be shown in each of the four quadrants and the logarithm graph had to be shown in the first and fourth quadrants. The third mark was earned if the curves were correct in the first quadrant and if the one point of intersection was indicated in some way. Some candidates did not earn this final mark because they failed to draw attention to the point of intersection.	
		ii	Refer, in some way, to graphs crossing x-axis at $x = 1$ and $x = 2$ and that intersection is between these values	B1	AG; the values 1 and 2 may be assumed from part (i) if clearly marked there; dependent on curves being (more or less) correct in first quadrant; carrying out the sign-change routine is B0 <u>Examiner's Comments</u>	

					Candidates were required here merely to mention the fact that the curves cross the *x*-axis at 1 and 2 and that the *x*-coordinate of the point of intersection lies between these two values. Many managed this but there were some lengthy and convoluted attempts as well. Candidates embarking on a sign change routine did not earn the mark.	
		iii	Obtain correct first iterate	B1	to at least 3 dp (except in the case of starting value 1 leading to 2)	
		iii	Show correct iterative process	M1	involving at least 3 iterates in all; may be implied by plausible converging values	
		iii	Obtain at least 3 correct iterates	A1	allowing recovery after error; iterates given to at least 3 dp; values may be rounded or truncated answer required to exactly 3 dp; answer only with no evidence of process is 0/4	
		iii	Conclude with 1.917	A1	<u>Examiner's Comments</u> Candidates had no difficulty with this part and 88% of them duly earned all four marks.	
		iv	$1 \to 2 \to 1.91139 \to 1.91731\ldots \to 1.91690\ldots \to 1.91693\ldots$ $1.5 \to 1.94865\ldots \to 1.91479\ldots \to 1.91707\ldots \to 1.91692\ldots$ $2 \to 1.91139\ldots \to 1.91731\ldots \to 1.91690\ldots \to 1.91693\ldots$			
		v	Obtain 3.92 or greater accuracy	B1√	following their answer to part (iii)	
		v	Attempt 4 × ln(part (iii) answer)	M1		
		v	Obtain *y*-coordinate 2.60	A1	value required to exactly 2 dp (so A0 for 2.6 and 2.603) <u>Examiner's Comments</u> This request proved much more challenging and 45% of candidates earned no marks. There were many attempts that involved finding the equations of the transformed curves; candidates seemed to hope that equating these would somehow reveal the ordinates of the new point of	

					intersection. Candidates adopting the appropriate course of tracking the transformation of the point of intersection did not always succeed. Many were correct in stating the *x*-coordinate as 3.92 but it was common then for the *y*-coordinate to be incorrectly given as 5.46, resulting from 4 ln 3.917.	
			Total	11		
2		i	State $y > 3$ or $f(x) > 3$ or $f > 3$ or 'greater than 3'	B1	must be > not ≥; allow $3 < y < \infty$ **Examiner's Comments** Only 40% of the candidates earned this mark. A quite common response was $y \geq 3$, which did not earn the mark, and other attempts involved 0 or 7.	
		ii	Obtain expression or eqn involving $\ln(\frac{y-3}{4})$ or $\ln(\frac{x-3}{4})$	M1	or equivs such as $\ln(\frac{4}{y-3})$ or $\ln(\frac{4}{x-3})$	
		ii	Obtain $\ln(\frac{4}{x-3})$ or $-\ln(\frac{x-3}{4})$	A1	or equiv	
		ii	State domain is $x > 3$ or equiv	B1FT	following answer to part **(i)** (but with adjustment so that reference is to *x*) **Examiner's Comments** Many candidates found the correct expression for the inverse function but there was much uncertainty about the domain and range. Some candidates ignored these two requests and many others found it difficult to determine the range or to express it clearly; a statement saying that the range is all real numbers was perfectly adequate.	
		ii	State range is all real numbers or equiv	B1		
		iii	Obtain correct first iterate	B1	showing at least 3 dp; B0 if initial value not 3 but then M1A1A1 available	

		iii	Show correct iteration process	M1	showing at least 3 iterates in all; may be implied by plausible converging values; M1available if based on equation with just a slip in $x = f(x)$ but M0 if based on clearly different equation	
		iii	Obtain at least 3 correct iterates	A1	allowing recovery after error; iterates to only 3 dp acceptable; values may be rounded or truncated	
		iii	Obtain (3.168, 3.168)	A1	each coordinate required to exactly 3 dp; award A0 if fewer than 4 iterates shown; part (iii) consisting of answer only gets 0 out of 4 [3 → 3.199148.. → 3.163187.. → 3.169162.. → 3.168155.. → 3.168324..] **Examiner's Comments** As in previous series, the iteration was usually carried out efficiently and accurately. However many candidates were guilty of not reading the question carefully and concluded with only $x = 3.168$, thereby losing the final mark.	
		iv	State P is point where the curves meet	B1	or equiv **Examiner's Comments** Most candidates did earn the mark but, in many cases, more information was provided than was necessary. There were many references to reflection or to the curves being symmetrical about the line $y = x$ but, provided that P was recognised as the point of intersection of the two curves, the mark was awarded.	
			Total	10		
3		i	Draw inverted parabola roughly symmetrical about the y-axis and with maximum point more or less on y-axis	M1	drawing enough of the parabola that two intersections occur, ignoring their locations at this stage	
		i	State $y = 9 - x^2$ and indicate two intersections by marks on diagram or written reference to two intersections	A1	now needs second curve drawn so that right-hand intersection occurs in first quadrant	

		ii	Calculate values of quartic expression for 2.1 and 2.2	M1	if no explicit working seen, M1 is implied by at least one correct value; but if no explicit working seen and both values wrong, award M0	
		ii	Obtain −1.9… and 1.6… and draw attention to sign change or clear equiv	A1		
		ii	Obtain correct first iterate	B1	starting anywhere between −1 and 9 and showing at least 3 d.p.	
		ii	Carry out process to produce at least three iterates in all	M1	implied by plausible sequence of values; allow recovery after error	2.1 → 2.15056 → 2.15531 → 2.15575 → 2.15579 2.15 → 2.15526 → 2.15574 → 2.15579
		ii	Obtain at least two more correct iterates	A1	showing at least 3 decimal places	
		ii	Obtain 2.156	A1	final answer needed to exactly 3 d.p.; not given for 2.156 as final iterate in sequence, i.e. needs indication (perhaps just underlining) that value of α found **Examiner's Comments** Part (i) was not answered well and only 23% of candidates earned two marks; candidates did need to show some care in order to earn the second mark. Many candidates drew a parabola with a minimum point, evidently believing that the curve $y = x^2 - 9$ was needed. Others realised that $y = 9 - x^2$ was required but the sketch did not show the maximum point anywhere near the y-axis. The curve given in the question is easily shown to cross the x-axis at the point (2, 0) and candidates were expected to show the second curve crossing the x-axis to the right of this point. Not all candidates drew attention to the two points of intersection and not all gave the equation of the second curve in an acceptable form involving variables x and y The remaining parts of this question were answered very well. The necessary calculations in part (a) were shown and reference was made to the sign change. The iterative process in part (b) was carried out efficiently although, in a minority of cases, correct values were followed by an incorrect conclusion of 2.155 or 2.16.	2.2 → 2.15980 → 2.15616 → 2.15583 → 2.15580 answer only: 0/4

			Total	8		
4		i	State or clearly imply $a=\frac{1}{2}$	B1	$a=\frac{5}{2}$ and $b=\frac{1}{2}$ earn B0 B0	
		i	State or clearly imply $b=\frac{5}{2}$ (Implied by, for example, just $\frac{1}{2}$ and $\frac{5}{2}$ stated in that order)	B1	$\sin(-\frac{1}{2}\pi)+\frac{3}{2}$ and $\sin(\frac{1}{2}\pi)+\frac{3}{2}$ earn B0 B0	
		i			**Examiner's Comments** There was evidence that some candidates were not particularly familiar with the inverse sine function in this question; manipulation and evaluation of expressions were not always carried out accurately. Answers to part (i) demonstrated this unease as some responses involved π or ±8. Many candidates were able to write down the two correct values of *a* and *b* immediately. Others identified the stretch and translation of the curve *y* = *sin⁻¹x* required to produce the given curve before determining the two values. Another approach involved solving two equations.	
		ii	**(a)** Carry out relevant calculations using radians	M1	Involving $8\sin^{-1}(x-\frac{3}{2})$ or $8\sin^{-1}(x-\frac{3}{2})-x$ or equiv; needs two explicit calculations	May carry out calculations in, for example, $\frac{3}{2}+\sin(\frac{1}{8}x)-x$
		ii	Obtain 1.6 and 2.4 or −0.1 and 0.6	A1	Or equivs	
		ii	Conclude with reference to 1.6 < 1.7 but 2.4 > 1.8, or to sign change	A1	Or equiv **Examiner's Comments** Part (ii)(a) is a routine verification of the location of the root but only 57% of the candidates earned all the marks. Some candidates used their calculators in degree mode and no marks were available in this part. A more significant problem concerned those candidates who evaluated $8\sin^{-1}(x-\frac{3}{2})$ at the two values. Obtaining 1.61 and 2.44, many were clearly surprised not to find a sign change; some then realised that subtraction of 1.7 and 1.8 respectively was needed or that	

					an observation that 1.61 < 1.7 whereas 2.44 > 1.8 was required. Evidence of calculations was needed; candidates who merely stated that there would be a sign change earned no marks in this part.	
		ii	**(b)** State or imply $p = \frac{3}{2}$ and $q = \frac{1}{8}$	B1	Implied by presence in iterative formula	
		ii	Obtain correct first iterate	B1	Having started with value x_1 such that $1.7 \le x_1 \le 1.8$; given to at least 4 s.f.	
		ii	Carry out iteration process	M1	Obtaining at least three iterates in all; having started with any non-negative value; implied by an apparently converging sequence of plausible values; all values to at least 4 s.f.	Answer only can earn no more than the first B1 for values of p and q; working in degrees can earn no more than the first B1 (for p and q) and M1
		ii	Obtain at least three correct iterates	A1	Allowing recovery after error	
		ii	Conclude with clear statement that root is 1.712	A1	Final answer required to exactly 4 significant figures <u>Examiner's Comments</u> Use of iteration to find a root is usually a good source of marks for candidates in this unit. But on this occasion, this was not always the case; only 55% of the candidates earned all the marks in part (ii)(b) and 21% recorded no marks. Some did not know how to set up the iterative formula or there were errors in establishing the values of p and q; it was not uncommon for $q = 8$ to be stated. There was limited credit available for those candidates using degrees. There was also confusion between significant figures and decimal places and those candidates offering 1.7124 as their final answer did not earn the final mark.	
			Total	10		
5		i	Draw more or less correct sketch of $y = \cos^{-1}x$ existing in first and second quadrants	*B1	Ignore any curve outside $0 \le y \le \pi$; condone no or wrong intercepts on axes	
		i	Draw U-shaped parabola passing through origin and showing minimum point	*B1	Curve must exist in first and third quadrants	

		i	Indicate one intersection in first quadrant by blob or reference in words or …	B1	Dep *B *B	
		ii	Obtain correct first iterate showing at least 4 s.f. rounded or truncated	B1		
		ii	Show iterative process to produce at least three iterates in all showing at least 3 s.f.	M1	Implied by incorrect values apparently converging	0.25 0.23965… 0.24250… 0.24172… 0.24193…
		ii	Obtain at least four correct iterates in all showing at least 4 s.f.	A1	Allowing recovery after error	
		ii	Conclude with value 0.242	A1	Answer to be clearly indicated by underlining final value in sequence or by separate statement; answer required to precisely 3 s.f.; allow final A1 even if iterates have been shown to only 3 s.f.; answer only earns 0/4	
		iii	State $y = -\cos^{-1}(-x)$ or $y = \cos^{-1}x - \pi$	B1		
		iii	State $y = x(-2x + 5)$ or equiv	B1	Allow $y = - - x(2(-x) + 5)$ or similar; condone missing $y =$ in each case	
		iii	State -0.242 for x-coordinate	B1 FT	Following their answer to (**ii**); allow greater accuracy here	
		iii	State -1.33 for y-coordinate	B1	Allow value rounding to -1.33 **Examiner's Comments** The response to part (i) was disappointing and 55% of the candidates earned no more than 1 of the 3 marks available. For the first sketch, many candidates seemed content to draw any parabola and there were many attempts consisting of a parabola with its minimum point at the origin. Great accuracy is not required for a sketch but the essential features are expected to be there; in this case the mark was earned for a curve passing through the origin and showing a minimum point in the third quadrant. It was clear that many candidates were unfamiliar with the inverse cosine curve; there were many attempts at the curve $y = \sec x$. For those candidates with the right idea, many did not restrict their attempts to the principal values. Provided the curve was reasonable for the values $0 \le y \le \pi$, the mark was not lost if the curve extended beyond these values. The third mark was available for	

					indicating in some way that the single intersection of the two curves meant exactly one real root; not all candidates with acceptable curves earned this third mark. There was generally confident work using radians in answering part (ii) and 89% of the candidates earned at least 3 of the 4 marks available. A few candidates made a slip in applying the iterative formula or tried to use a completely different formula. Some did not follow the directions in the question, either giving the values in the sequence correct to only 3 significant figures or, more usually, giving the value of the root as 0.2419. Part (iii) presented several problems and only 27% of the candidates earned all 4 marks. Many candidates struggled to keep track of various minus signs when giving the equation of the transformed parabola. There was greater success with the other curve and the correct equation $y = -\cos^{-1}(-x)$ was the common response. Some candidates went no further with their attempts at this part and many others evidently expected to be able to solve the equation formed by equating the equations of their new curves. Candidates who were aware of the geometrical aspects of the question realised that the pair of transformations could be applied to the point of intersection of the two original curves. They had no trouble in identifying the required x-coordinate as −0.242 although there were some errors in calculating the corresponding y-coordinate.	
			Total	11		
6		a	$f'(x) = 3x^2 - 2x - 5$ $$x_{n+1} = x_n - \frac{x_n^{\ 3} - x_n^{\ 2} - 5x_n + 10}{3x_n^{\ 2} - 2x_n - 5}$$	B1(AO1.1) M1(AO1.1) E1(AO2.1)	Substitute into correct formula for Newton-Raphson	

			$x_{n+1} = \dfrac{3x_n^3 - 2x_n^2 - 5x_n - (x_n^3 - x_n^2 - 5x_n + 10)}{3x_n^2 - 2x_n - 5} =$ $= \dfrac{2x_n^3 - x_n^2 - 10}{3x_n^2 - 2x_n - 5}$	[3]	AG a correct intermediate step leading to the given answer is required	
		b	$x_2 = -2.607$ $x_3 = -2.535$ $x_4 = -2.533$	B1(AO1.1) [1]	BC All three values must be given to 4 significant figures.	
		c	f (–2.5325) and f (–2.5335) $(-2.5325)^3 - (-2.5325)^2 - 5(-2.5325) + 10 = 0.0066125$ $(-2.5335)^3 - (-2.5335)^2 - 5(-2.5335) + 10 = -0.0127017$ Since f (–2.5325) > 0 and f (–2.5335) < 0 x_4 is α to 4 s.f.	M1(AO1.1) A1(AO2.1) E1(AO2.4) [3]	Accept other alternative values which would confirm α as a root correct to 4 s.f. At least the result of evaluation must be shown The change of sign must be pointed to	
		d	$3(-1)^2 - 2(-1) - 5 = 0$ Since the fraction is undefined at $x = -1$, x_2 is undefined	B1(AO2.1) E1(AO1.2) [2]	Accept references to a stationary point of the function | or the tangent to the curve being horizontal	
			Total	9		

7		a	Draw (more or less) correct sketch of $y = \dfrac{5}{x^2}$ Draw (more or less) correct sketch of $y = \|2x - 4\|$ Indicate two points of intersection	B1(AO1.1) B1(AO1.1) B1(2.2a) [3]	Must have both branches, and axes clearly shown as asymptotes V shape with vertex on the positive *x*-axis and intersecting the positive *y*-axis With both sketches correct | *x*- and *y*-intercept values need not be stated	
		b	$\dfrac{5}{x^2} = -(2-4x) \Rightarrow \mathrm{f}(x) = 4x^3 - 2x^2 - 5$	B1(AO3.1a) [1]	AG	
		c	$x_{n+1} = x_n - \dfrac{4x_n^3 - 2x_n^2 - 5}{12x_n^2 - 4x_n}$ $= \dfrac{12x_n^3 - 4x_n^2 - 4x_n^3 + 2x_n^2 + 5}{12x_n^2 - 4x_n}$ $= \dfrac{8x_n^3 - 2x_n^2 + 5}{12x_n^2 - 4x_n}$	M1(AO1.1) M1(AO1.1) E1(AO2.1) [3]	Correct derivative seen and substituted into correct N-R formula Combining terms; either one single fraction seen or two fractions with a common denominator **AG**; working as in	

					line above must be seen	Suffices must be present
		d	$x_2 = \frac{11}{8},\ x_3 = 1.28090\ldots$ $x_4 = 1.27232\ldots,\ x_5 = 1.27225\ldots$ $\alpha = 1.272$	B1(AO1.1) B1(AO1.1) B1(AO1.1) [3]		
		e	$y = 4x - 2$ was used to obtain the N-R formula and this line only intersects the 1st quadrant branch so can only give the positive root	B1(AO2.4) [1]		
			Total	11		
8		a	1.4422, 1.5099, 1.5197, 1.5211, 1.5213, 1.5214… Hence $\alpha = 1.521$	B1(AO1.1a) M1(AO1.1) A1(AO2.2a) [3]	Correct x_2 Use correct iterative process Obtain 1.521 (must be 4sf)	
		b	$F'(x) = -(x^2 + 4x)x^{-4}$ $F'(\alpha) = -1.57$ Will only converge if $\lvert F'(\alpha) \rvert < 1$	B1(AO1.1) B1ft(AO1.1) E1(AO1.2)	Correct $F'(x)$ Correct $F'(\alpha)$	

				[3]	Identify correct condition	Follow their value of α
			Total	6		
9	a	(i)	$f'(x) = \dfrac{3}{3x+1} - 1$	B1 (AO 1.1)	Correct $f'(x)$	
			$\dfrac{3}{3x+1} - 1 = 0$	M1 (AO 2.1)	Equate to 0 and solve for x	Must be seen
			$x = \frac{2}{3}$	A1 (AO 1.1) [3]	Obtain $x = \frac{2}{3}$	
		(ii)	E.g. The tangent at α will be parallel to the x-axis so will never intersect the x-axis to give x_2.	E1 (AO 2.4) [1]	Correct reason referring to the tangent at α	A sketch is allowable if annotated.
	b		$x_{n+1} = x_n - \dfrac{\ln(3x_n+1) - x_n}{\dfrac{3}{3x_n+1} - 1}$	B1 (AO 2.1)	State correct N-R formula	
			$x_{n+1} = x_n - \dfrac{(3x_n+1)(\ln(3x_n+1) - x_n)}{3-(3x_n+1)}$	M1 (AO 2.1)	Attempt rearrangement	Allow no subscripts

			$x_{n+1} = \frac{x_n(2-3x_n)-(3x_n+1)\ln(3x_n+1)+x_n(3x_n+1)}{2-3x_n}$ $x_{n+1} = \frac{2x_n-3x_n{}^2-(3x_n+1)\ln(3x_n+1)+3x_n{}^2+x_n}{2-3x_n}$ $x_{n+1} = \frac{3x_n-(3x_n+1)\ln(3x_n+1)}{2-3x_n}$ AG	A1 (AO 2.5) [3]	Obtain given N-R formula	
		c	x_2 = 2.54518 x_3 = 1.94865, x_4 =1.90416, x_5 = 1.90381, x_6 = 1.90381 β = 1.9038	B1 (AO 1.1a) M1 (AO 1.1a) A1 (AO 1.1) [3]	Correct first iterate Use correct iterative process to find at least two further values Obtain β = 1.9038	At least 4sf At least 4sf From at least 5 iterates, to at least 5sf
		d	ln(3 × 1.90375 + 1) − 1.90375 = 0.000035223 ln(3 × 1.90385 + 1) − 1.90385 = −0.000020077 f(1.90375) > 0 and f(1.90385) < 0	M1 (AO 2.1) A1* (AO 2.1)	Attempt values either side of root Obtain both correct values	Allow other valid methods

			so by sign change $1.90375 < \beta < 1.90385$ so must 1.9038 correct to 5sf	dep*A1 (AO 2.4)	Must refer to sign change	
			Total	13		
10		a	At A, $y = 0$ so $-\tan^{-1}\left(\frac{1}{2}x - \frac{1}{3}\pi\right) = 0$ $\frac{1}{2}x - \frac{1}{3}\pi = 0$ $x = \frac{2}{3}\pi$ so A is $\left(\frac{2}{3}\pi, 0\right)$ At B, $x = 0$ so $y = 0.808$ so B is (0, 0.808)	M1 (AO 1.1a) A1 (AO 1.1) B1 (AO 1.1) [3]	Attempt x-coordinate at A Obtain $\left(\frac{2}{3}\pi, 0\right)$ Obtain (0, 0.808), or better Allow decimal equiv 2.094…	
		b	reflection in the x-axis translation in the x direction by $\frac{1}{3}\pi$ then stretch in x direction by sf 2	B1 (AO 1.1) M1 (AO 1.1a) A1 (AO 2.5) [3]	Stated at any point Translation by $\pm\frac{1}{3}\pi$ and stretch by sf 2 or $\frac{1}{2}$, both in the x direction Allow informal language for the M1 Must use 'factor' or 'scale factor' Allow stretch then translation, as long as details are commensurate with order	

		c	$0 < 0.808$ $1 > 0.501$ change in inequality sign hence $0 < \text{root} < 1$	M1 (AO 2.1) E1 (AO 2.4) [2]	Substitute $x = 0$ and $x = 1$ into both sides of the equation Conclude appropriately	Could also rearrange to $\mathrm{f}(x) = x + \tan^{-1}\left(\frac{1}{2}x - \frac{1}{3}\pi\right) = 0$ and attempt f(0) = –0.808 and f(1) = 0.499 Refer to change in sign	
		d	eg $x_1 = 0.5$, $x_2 = 0.6730$ 0.6179, 0.6360, 0.6301, 0.6320, 0.6314, 0.6316, 0.6315, 0.6315… hence root is 0.632	B1 (AO 1.1a) M1 (AO 1.1a) A1 (AO 1.1) [3]	Correct first iterate for $0 < x < 1$ Attempt correct iterative process Obtain root as 0.632	At least 2 more values Must be 3sf	
		e	add $y = x$ to diagram in P.A.B. and show first iteration at least 4 more lines to show cobweb	M1 (AO 1.2) A1 (AO 1.2) [2]	Vertical line from x_1 and horizontal line to $y = x$ ie 2 vertical and 2 horizontal lines		
			Total	13			
11		a	$\frac{1}{e^{2x}} = \frac{x}{x^2+3} \Rightarrow x^2 + 3 = xe^{2x}$ $x^2 + 3 = xe^{2x} \Rightarrow x^2 + 3 - xe^{2x} = 0$	M1 (AO 1.1) A1 (AO 2.2a)	Equate expressions and cross-multiply (to remove fractions) **AG** – sufficient working must be		

				[2]	shown to indicate that result has been derived correctly	
		b	$\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{2x}) = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$ $h'(x) = 2x - \mathrm{e}^{2x} - 2x\mathrm{e}^{2x}$ $x_{n+1} = x_n - \frac{x_n^2 + 3 - x_n\mathrm{e}^{2x_n}}{2x_n - \mathrm{e}^{2x_n} - 2x_n\mathrm{e}^{2x_n}}$ $x_{n+1} = \frac{2x_n^2 - x_n\mathrm{e}^{2x_n} - 2x_n^2\mathrm{e}^{2x_n} - x_n^2 - 3 + x_n\mathrm{e}^{2x_n}}{2x_n - \mathrm{e}^{2x_n}(1+2x_n)}$ $x_{n+1} = \frac{x_n^2 - 2x_n^2\mathrm{e}^{2x_n} - 3}{2x_n - \mathrm{e}^{2x_n}(1+2x_n)} = \frac{x_n^2(1-2\mathrm{e}^{2x_n}) - 3}{2x_n - (1+2x_n)\mathrm{e}^{2x_n}}$	M1 (AO 1.1) A1 (AO 1.1) M1* (AO 2.1) M1dep* (AO 1.1) A1 (AO 2.2a) [5]	Attempt at product rule for $x\mathrm{e}^{2x}$ – expression must be of the form $\pm\mathrm{e}^{2x}$ (1 ± kxe^{2x}) Correct application of NR with their $h'(x)$ Correctly combining as a single fraction and expanding any brackets in numerator **AG** – sufficient working must be shown as the answer is given	
		c	From graph eg $x_1 = 1$ $x_2 = 0.83195181\ldots$, $x_3 = 0.7754682\ldots$	M1 (AO 3.1a) A1 (AO 1.1) A1 (AO 1.1)	Suitable starting value chosen	

			$x_4 = 0.77016\ldots$, $x_5 = 0.77011\ldots$ $\alpha = 0.770$ (correct to 3 dp)	[3]	At least two correct applications of NR	
	d		DR $$\text{fg}(x) = \text{f}\left(\text{e}^{-2x}\right) = \frac{\text{e}^{-2x}}{\left(\text{e}^{-2x}\right)^2 + 3}$$ $2\text{e}^{-4x} - 13\text{e}^{-2x} + 6 = 0$ $k = \text{e}^{-2x}$ $(2k-1)(k-6) = 0$ $$\text{e}^{-2x} = \tfrac{1}{2} \Rightarrow x = -\tfrac{1}{2}\ln\left(\tfrac{1}{2}\right)$$ $\text{e}^{-2x} \neq 6$Q it is given that $x \geq 0$	M1 (AO 2.1) A1 (AO 1.1) M1* (AO 3.1a) M1dep* (AO 1.1) A1 (AO 2.2a) A1 (AO 2.3) [6]	Attempt at fg(x) – need not be simplified Correct equation – fractions must be removed and powers simplified Substitute for e^{-2x} (or equivalent) Attempt to solve resulting quadratic www oe Correct statement or equivalent that e^{-2x} cannot be greater than 1	Or equivalent **Alternatively:** Factorise into two brackets containing e^{-2x} M2
			Total	16		