1. The positive constant a is such that $\int_a^{2a} \frac{2x^3 - 5x^2 + 4}{x^2}\,dx = 0$.

i. Show that $3a^3 - 5a^2 + 2 = 0$.

[6]

ii. Show that $a = 1$ is a root of $3a^3 - 5a^2 + 2 = 0$, and hence find the other possible value of a, giving your answer in simplified surd form.

[6]

2. The cubic polynomial $f(x)$ is defined by $f(x) = x^3 - 19x + 30$.

i. Given that $x = 2$ is a root of the equation $f(x) = 0$, express $f(x)$ as the product of 3 linear factors.

[4]

ii. Use integration to find the exact value of $\int_{-5}^{3} f(x)\,dx$.

[4]

iii. Explain with the aid of a sketch why the answer to part **(ii)** does not give the area enclosed by the curve $y = f(x)$ and the x-axis for $-5 \leqslant x \leqslant 3$.

[2]

3. (a) Find $\int (x^3 - x^2 - 2x)\,dx$. [3]

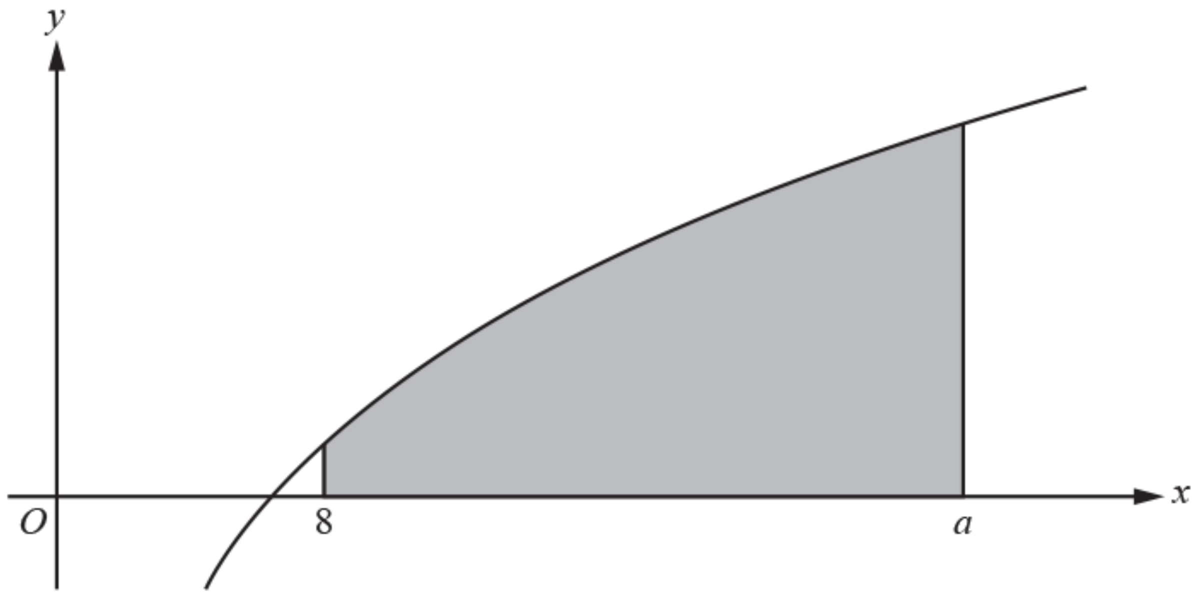
In this question you must show detailed reasoning.

(b)

Find the area enclosed by the curve $y = x^3 - x^2 - 2x$ and the positive x-axis. [4]

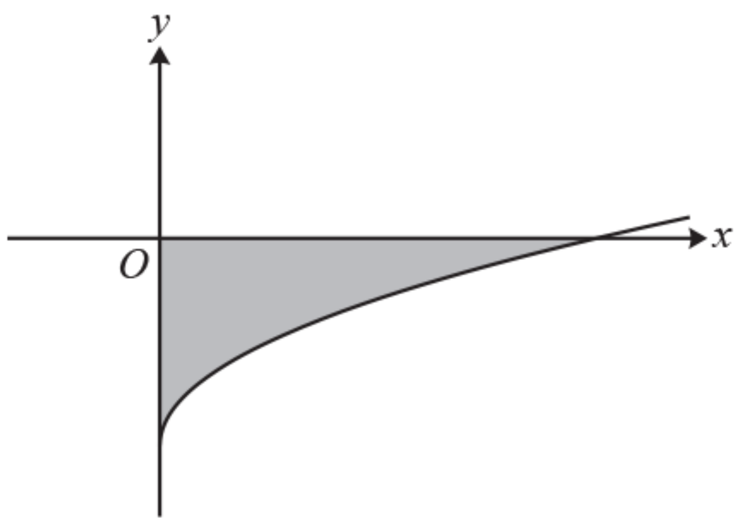
4. **In this question you must show detailed reasoning.**

The diagram shows part of the graph of $y = 2x^{\frac{1}{3}} - \frac{7}{x^{\frac{1}{3}}}$. The shaded region is enclosed by the curve, the *x*-axis and the lines $x = 8$ and $x = a$, where $a > 8$.



Given that the area of the shaded region is 45 square units, find the value of *a*. [9]

5.



The diagram shows the curve $y = \sqrt{x} - 3$. The shaded region is bounded by the curve and the two axes.

Find the exact area of the shaded region. [4]

END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Part marks and guidance	
1		i	$\int(2x - 5 + 4x^{-2})\,dx = x^2 - 5x - 4x^{-1}$	M1	Attempt to rewrite integrand in a suitable form	Attempt to divide all 3 terms by x^2, or attempt to multiply all 3 terms by x^{-2} soi
		i	$(4a^2 - 10a - {}^2/_a) - (a^2 - 5a - {}^4/_a) = 0$	A1	Obtain $2x - 5 + 4x^{-2}$	Allow if third term is written in fractional form
		i	$3a^2 - 5a + {}^2/_a = 0$ $3a^3 - 5a^2 + 2 = 0$ **AG**	M1	Attempt integration of their integrand	Their integrand must be written as a polynomial ie with all terms of the form kx^n, and no brackets At least two terms must increase in power by 1 Allow if the – 5 disappears
		i		A1	Obtain $x^2 - 5x - 4x^{-1}$	Allow unsimplified (eg ${}^4/_{-1}x^{-1}$)
		i		M1	Attempt use of limits	Must be F(2a) – F(a) ie subtraction with limits in the correct order Allow if no brackets ie $4a^2 - 10a - {}^2/_a - a^2 - 5a - {}^4/_a$ Must be in integration attempt, but allow M1 for limits following M0 for integration eg if fraction not dealt with before integrating
		i		A1	Equate to 0 and rearrange to obtain $3a^3 - 5a^2 + 2 = 0$	Must be equated to 0 before multiplying through by a At least one extra line of working required between $(4a^2 - 10a - {}^2/_a) - (a^2 - 5a - {}^4/_a) = 0$ and the final answer **AG** so look carefully at working <u>Examiner's Comments</u> The quality of responses to this question varied considerably. Not knowing how to deal with the

						rational expression proved to be a stumbling block for many candidates, resulting in flawed integration attempts. These candidates could still gain a mark for attempting to use limits correctly in their integral, and many did gain this mark though some went back to using the original function. Candidates who appreciated the need to first rewrite the integrand usually did so successfully. Dividing each term by x^2 tended to be the more popular and successful approach. Some attempted to multiply through by x^2 which gave the correct final term, but errors in applying rules of indices sometimes led to the first two terms being wrong. The integration was usually carried out correctly on whatever function they had at this stage, as was the attempted use of limits. Of the candidates who had been successful up to this point, fewer than half were then able to show the given equation correctly. Giving the first term as $2a^2$ rather than $(2a)^2$ was a common error, and substituting values into $4x^{-1}$ caused difficulties for many; those who first rewrote this term in fractional form were usually more successful.
		ii	$f(1) = 3 - 5 + 2 = 0$ **AG** $f(a) = (a - 1)(3a^2 - 2a - 2)$ $a = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$ hence $a = \frac{1}{3}(1 + \sqrt{7})$	B1	Confirm $f(1) = 0$ – detail required	**Allow working in x not a throughout** $3(1)^3 - 5(1)^2 + 2 = 0$ is enough B0 for just $f(1) = 0$ If using division must show '0' on last line If using coefficient matching must show 'R = 0' If using inspection then there must be some indication of no remainder eg expand to show correct cubic
		ii		M1	Attempt full division by $(a - 1)$, or equiv method	Must be complete method – ie all 3 terms attempted

						Long division – must subtract lower line (allow one slip) Inspection – expansion must give at least three correct terms of the cubic Coefficient matching – must be valid attempt at all coeffs of quadratic, considering all relevant terms each time
		ii		A1	Obtain $3a^2$ and one other correct term	Could be middle or final term depending on method Must be correctly obtained Coeff matching – allow for $A = 3$ etc
		ii		A1	Obtain fully correct quotient	Could appear as quotient in long division, or as part of a product if using inspection. For coeff matching it must now be explicit not just $A = 3$, $B = -2$, $C = -2$
		ii		M1	Attempt to solve quadratic	Using the quadratic formula, or completing the square (see guidance sheet) though negative root may be lost at any point M0 if factorising attempt as expected root is a surd Quadratic must come from division attempt, even if this was not good enough for first M1
		ii		A1	Obtain $\frac{1}{3}(1 + \sqrt{7})$ only	Must give the positive root only, so A0 if negative root still present (but condone $a = 1$ also given) Allow aef but must be a simplified surd as per request on question paper (ie simplify $\sqrt{28}$) **Examiner's Comments** The majority of candidates chose to use the factor theorem to confirm that $a = 1$ was a root, and this was invariably correct. Other methods were also used, but some candidates lost an easy mark by

						failing to address this part of the question. Finding the quotient was done well, although the absence of a linear term in the function caused problems for some, especially those using long division. There was a variety of methods used to find the quotient, including inspection. This method is fine if done successfully but it makes gaining partial credit extremely unlikely as no working is shown. Candidates attempting this method would be well advised to expand their brackets to check for accuracy. Those who had the correct quotient could usually solve the resulting equation correctly, although completing the square was not as successful as using the formula. The surd was invariably simplified correctly, but only a small minority of candidates referred back to the question and appreciated the need to reject the negative solution.
			Total	12		
2		i	$f(x) = (x - 2)(x^2 + 2x - 15)$	B1	State or imply that $(x - 2)$ is a factor	Could be stated explicitly, or implied by using it in an attempt at the quotient or a factorisation attempt Could also give $(2 - x)$ as the factor
		i		M1	Attempt complete division, or equiv	Must be dividing by $(x - 2)$, or by one of the two other correct factors (or the negative of any of these factors) No need to show zero remainder as told that $x = 2$ is a root Must be complete method – ie all 3 terms attempted Long division – must subtract lower line (allow one slip) Inspection – expansion must give at least three correct terms of the cubic Coefficient matching – must be valid attempt at all coeffs of quadratic, considering all relevant terms

						each time Synthetic division – must be using 2 (not −2) and adding within each column (allow one slip); expect to see $\begin{array}{c\|cccc} 2 & 1 & 0 & -19 & 30 \\ & & 2 & 4 & \\ \hline & 1 & 2 & -15 & \end{array}$
		i		A1	Obtain correct quotient of $x^2 + 2x - 15$ CWO	Or correct quotient for their factor Could be stated explicitly, seen in division attempt or implied by $A = 1$, $B = 2$, $C = -15$
					Obtain $(x - 2)(x + 5)(x - 3)$	
		i	$= (x - 2)(x + 5)(x - 3)$	A1	<u>Examiner's Comments</u> This proved to be a straightforward question for many candidates, and the majority gained full credit. A variety of methods continue to be seen when factorising a cubic, and inspection is becoming increasingly common. Coefficient matching continues to be seen as a routine	Must be written as a product of the three linear factors Allow any equiv eg $(2 - x)(x + 5)(3 - x)$ Full credit for repeated use of factor theorem, or just writing down correct product Ignore any subsequent reference to roots **SR** A fully correct factorisation resulting from division by $(x + 5)$ or $(x - 3)$ can still get full credit, even though the root of $x = 2$ was not used

					method that candidates of all abilities can employ successfully. More candidates are attempting to use algebraic long division, but errors tend to be more common as some candidates can be confused as to whether to add or subtract within the division. The lack of an x^2 term also caused problems for some. It was clear within some solutions that an alternative method had been attempted when the initial one failed. When this is the case candidates should ensure that they delete any working that does not form part of their final solution.	
		ii	$\left[\frac{1}{4}x^4 - \frac{19}{2}x^2 + 30x\right]_{-5}^{3}$	M1*	Attempt integration	Increase in power by 1 for at least 2 terms
		ii		A1	Obtain correct integral	Could also have + c present; condone dx or ∫ still present
		ii	= 24.75 – (– 231.25)	M1d*	Attempt correct use of limits	Must be F(3) – F(–5) Must be attempting the value of the requested definite integral, so M0 if instead attempting area (ie using x = 2 as a limit)
		ii	= 256	A1	Obtain 256 **Examiner's Comments** The integration attempt was invariably correct, and most candidates were able to attempt the correct use of limits. However, evaluating an expression involving negative numbers once again caused problems for a significant minority of candidates and it was relatively common for F(-5) to be incorrect. As long as there was evidence of the	A0 for 256 + c Answer only is 0/4 – need to see evidence of integration, but use of limits does not need to be explicit

					limits having been correctly attempted then candidates were awarded the method mark for this. However if candidates simply write down an incorrect numerical evaluation, with no evidence to support this, then examiners cannot speculate as to what may or may not have been attempted, and no credit can be awarded.	
		iii	Sketch positive cubic with 3 distinct roots	B1	Sketch f(x) for $-5 \le x \le 3$	Must be a positive cubic Allow if maximum point is on y-axis No need for roots to be labelled, but need one negative and two positive roots (or ft from an incorrect factorisation in (i) – could have fewer than 3 roots shown if this is consistent with their roots in required range) Graph must be sketched for at least $-5 \le x \le 3$, but it is fine if more is shown – only penalise explicitly incorrect roots
		iii	Some of the area is below the x-axis which will make negative contribution to the total	B1	Explanation referring to the area below the x-axis giving a negative value	B0B1 is possible (including following no sketch at all) Need to mention 'negative' and identify the relevant area in some way eg 'below x-axis' or $2 \le x \le 3$ or clear shading Just referring to some area below x-axis is insufficient, as is any reference just to negative area B0 for statements indicating that some area is ignored / cannot be calculated within an otherwise correct statement A reason is required as to why (ii) is incorrect – it is not sufficient to just state that the actual area is larger, or to just describe how to find the area

					Examiner's Comments For the first mark candidates were expected to provide a sketch of the cubic and many were able to do so, though there was a disappointing lack of care in the sketches. For the second mark candidates had to demonstrate an appreciation that an area under the *x*-axis would give a negative result when found by integration. Expressing this idea clearly proved to be beyond the capabilities of many of the candidates. Many of the answers could identify that it was the region below the *x*-axis that was at issue, but they failed to explain what the issue was. A number of solutions stated that this region would be ignored in the integration, and others used shading to show that 'the area under the curve' would be infinite for this region. Some solutions referred to 'areas cancelling out', but lacked detail or precision, and others described how the integration should be done without explaining why this method had failed. Whilst a number of concise and detailed explanations were seen, it was disappointing that so many candidates were unable to express their reasoning with the clarity required.	
			Total	10		
3		a	$\frac{x^4}{4}-\frac{x^3}{3}-x^2$ $\frac{x^4}{4}-\frac{x^3}{3}-x^2+c$	M1(AO1.1) A1(AO1.1) A1(AO1.2) [3]	Increase at least two indices by 1 At least 2 terms correct	

					All correct with + c	
		b	DR $x^3 - x^2 - 2x = 0$ $x = 0$ or 2 (or -1) $\int_0^2 (x^3 - x^2 - 2x)\,dx$ $(= -\frac{8}{3})$ Hence area $= \frac{8}{3}$	M1(AO1.1a) A1(AO1.1) M1(AO1.1) A1(AO2.2a) [4]	Allow just $x = 0$ or 2	
			Total	7		
4			DR $\int_8^a 2x^{\frac{1}{3}} - 7x^{-\frac{1}{3}}\,dx = 45$ $\left[\frac{2x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{7x^{\frac{2}{3}}}{\left(\frac{2}{3}\right)}\right]_8^a (= 45)$	M1* (AO 3.1a) A1 (AO 1.1) A1 (AO 1.1) Dep*M1 (AO 1.1)	If $a = 27$ with no working then 0/9 M1 – attempt integration (increase in power by 1 for at least 1 term) A1 – 1 term correct (accept unsimplified) A1 – both	

			$\frac{3}{2}a^{\frac{4}{3}}-\frac{21}{2}a^{\frac{2}{3}}-\left(\frac{3}{2}(8)^{\frac{4}{3}}-\frac{21}{2}(8)^{\frac{2}{3}}\right)(=45)$ $\frac{3}{2}a^{\frac{4}{3}}-\frac{21}{2}a^{\frac{2}{3}}-(24-42)(=45)$ $a^{\frac{4}{3}}-7a^{\frac{2}{3}}-18=0$ $\left(a^{\frac{2}{3}}-9\right)\left(a^{\frac{2}{3}}+2\right)=0$ $a^{\frac{2}{3}}=9\ \left(\text{and } a^{\frac{2}{3}}=-2\right)$ a = 27 only	A1 (AO 1.1) M1 (AO 1.1) M1 (AO 3.1a) A1 (AO 1.1) A1 (AO 2.2a) [9]	<table><tr><td>correct (accept unsimplified) *F*(*a*) – *F*(8) oe Equate integrated expression to 45 – dependent on both previous M marks Attempt to solve quadratic in $a^{\frac{2}{3}}$</td><td>SC if M0 for fourth M mark then award <table><tr><td>B1 for</td><td>$a^{\frac{2}{3}}=9$</td></tr></table> B1 *a* = 27 only</td></tr></table> <u>Examiner's Comments</u>	

					There were some encouraging attempts at this question, with the integration done accurately. Sign errors sometimes lead to the wrong quadratic, but many realised there was a ‘hidden’ quadratic. This question asked for detailed reasoning so we expected to see some working to show how the quadratic solutions were obtained, either factors or some working with the formula or completing the square. A few candidates found $F(8) - F(a)$ when integrating from 8 to a and some integrated from a to 8 from the start. Those who obtained $a = 27$ from their calculator with no justification/detailed reasoning scored zero, although if they checked by showing the integral from 8 to 27 was equal to 45, then the first three marks were possible. There was a significant minority of candidates that attempted to differentiate rather than integrate to find the area under the curve.	
			Total	9		
5			$\sqrt{x} - 3 = 0$ $x = 9$ $\int_0^9 (\sqrt{x} - 3)\mathrm{d}x$ $= -9$ Area = 9	B1 (AO 3.1a) M1 (AO 1.1) A1 (AO 1.1) A1FT (AO 3.2a) [4]	BC ft their negative result	

			Total	4	

