Numerical Methods

Questions

Q1.

The temperature, θ°C, of coffee in a cup, *t* minutes after the cup of coffee is put in a room, is modelled by the differential equation

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

where *k* is a constant.

The coffee has an initial temperature of 80 °C

Using *k* = 0.1

(a) use two iterations of the approximation formula $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_0}{h}$ to estimate the temperature of the coffee 3 minutes after it was put in the room.

(6)

The coffee in a different cup, which also had an initial temperature of 80 °C when it was put in the room, cools more slowly.

(b) Use this information to suggest how the value of k would need to be changed in the model.

(1)

(Total for question = 7 marks)

Q2.

Use Simpson's rule with 4 intervals to estimate

$$\int_{0.4}^{2} e^{x^2}\,dx$$

(Total for question = 5 marks)

Q3.

A population of deer was introduced onto an island.

The number of deer, P, on the island at time t years following their introduction is modelled by the differential equation

$$\frac{dP}{dt} = \frac{P}{5000}\left(1000 - \frac{P(t+1)}{6t+5}\right) \qquad t > 0$$

It was estimated that there were 540 deer on the island six months after they were introduced.

Use **two** applications of the approximation formula $\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h}$ to estimate the number of deer on the island 10 months after they were introduced.

(Total for question = 7 marks)

Q4.

Julie decides to start a business breeding rabbits to sell as pets.

Initially she buys 20 rabbits. After t years the number of rabbits, R, is modelled by the differential equation

$$\frac{dR}{dt} = 2R + 4\sin t \qquad t > 0$$

Julie needs to have at least 40 rabbits before she can start to sell them.

Use two iterations of the approximation formula

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

to find out if, according to the model, Julie will be able to start selling rabbits after 4 months.

(Total for question = 7 marks)

Q5.

The variables *x* and *y* satisfy the differential equation

$$\frac{d^2y}{dx^2} = 2y^2 - x - 1$$

where $\frac{dy}{dx}$ = 3 and *y* = 0 at *x* = 0

Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{(y_{n+1} - 2y_n + y_{n-1})}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_{n-1})}{2h}$$

with *h* = 0.1 to find an estimate for the value of *y* at *x* = 0.2

(Total for question = 7 marks)

Q6.

A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3}x \tanh t \qquad \text{(I)}$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula

$$\left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_n)}{h}$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(4)

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation

$$\frac{du}{dt} + u \tanh t = 1 + \frac{3}{\cosh t} \qquad \text{(II)}$$

(3)

(c) Determine the general solution of equation (II)

(4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied.

(3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate.

(3)

(Total for question = 17 marks)

Q7.

The variables *x* and *y* satisfy the differential equation

$$\frac{d^2y}{dx^2} + 15\frac{dy}{dx} - 3y^2 = 2x$$

where $y = 1$ at $x = 0$ and where $y = 2$ at $x = 0.1$

Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{(y_{n+1} - 2y_n + y_{n-1})}{h^2} \text{ and } \left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_{n-1})}{2h}$$

with $h = 0.1$ to find an estimate for the value of *y* when $x = 0.3$

(Total for question = 6 marks)

Q8.

The velocity v ms^{-1}, of a raindrop, *t* seconds after it falls from a cloud, is modelled by the differential equation

$$\frac{dv}{dt} = -0.1 v^2 + 10 \qquad t \geq 0$$

Initially the raindrop is at rest.

(a) Use two iterations of the approximation formula $\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h}$ to estimate the velocity of the raindrop 1 second after it falls from the cloud.

(5)

Given that the initial acceleration of the raindrop is found to be smaller than is suggested by the current model,

(b) refine the model by changing the value of one constant.

(1)

(Total for question = 6 marks)

Q9.

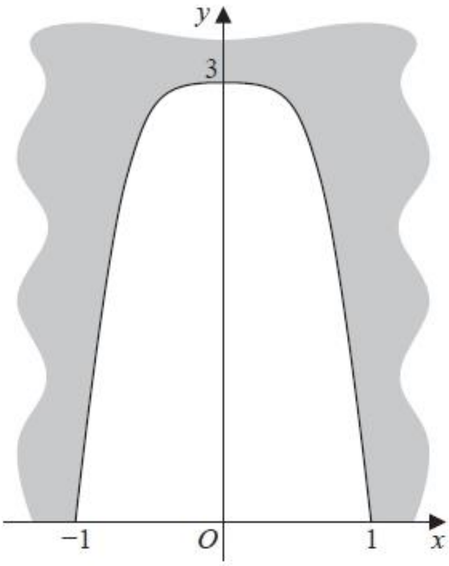

Figure 1

Figure 1 shows a sketch of the vertical cross-section of the entrance to a tunnel. The width at the base of the tunnel entrance is 2 metres and its maximum height is 3 metres.

The shape of the cross-section can be modelled by the curve with equation $y = f(x)$ where

$$f(x) = 3\cos\left(\frac{\pi}{2}x^2\right) \qquad x \in [-1, 1]$$

A wooden door of uniform thickness 85 mm is to be made to seal the tunnel entrance.

Use Simpson's rule with 6 intervals to estimate the volume of wood required for this door, giving your answer in m^3 to 4 significant figures.

(Total for question = 6 marks)

Q10.

The value, *V* hundred pounds, of a particular stock *t* hours after the opening of trading on a given day is modelled by the differential equation

$$\frac{dV}{dt} = \frac{V^2 - t}{t^2 + tV} \qquad 0 < t < 8.5$$

A trader purchases £300 of the stock one hour after the opening of trading.

Use two iterations of the approximation formula $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ to estimate to the nearest £ the value of the trader's stock half an hour after it was purchased.

(6)

(Total for question = 6 marks)

Q11.

Use Simpson's Rule with 6 intervals to estimate

$$\int_1^4 \sqrt{1 + x^3}\, dx$$

(5)

(Total for question = 5 marks)

Mark Scheme – Numerical Methods

Q1.

Question	Scheme	Marks	AOs
	$\frac{d\theta}{dt} = -k(\theta - 20)$, k is a constant. $\theta_0 = 80$		
(a)	{Two iterations from $t = 0$ to $t = 3 \Rightarrow$} $h = 1.5$		
	Uses $h = 1.5$, $\theta_0 = 80$, $k = 0.1$ (condone $k = -0.1$) in a complete strategy to find a numerical expression for $\theta_1 = ...$	M1	3.1b
	$\{\theta_0 = 80, k = 0.1 \Rightarrow\}$ $\left(\frac{d\theta}{dt}\right)_0 = -0.1(80 - 20)$ $\{= -6\}$	M1	3.4
	$\left\{\frac{\theta_1 - 80}{1.5} = -6 \Rightarrow\right\}$ $\theta_1 = 80 + (1.5)(-6)$	M1	1.1b
	$\theta_1 = 71$	A1	1.1b
	$\{\theta_1 = 71 \Rightarrow\}$ $\left(\frac{d\theta}{dt}\right)_1 = -0.1(\text{"}71\text{"} - 20)$ $\{= -5.1\}$	M1	1.1b
	$\theta_2 = 71 + (1.5)(-5.1) = 63.35\,(°\text{C})$	A1	2.1
		(6)	
(b)	Decrease k to become a smaller positive value	B1	3.5c
		(1)	
			(7 marks)

Notes

(a)	
M1:	See scheme
M1:	Uses the model to evaluate the initial value of $\frac{d\theta}{dt}$ using $k = 0.1$ (condone $k = -0.1$) and the initial condition $\theta_0 = 80$
M1:	Applies the approximation formula with $\theta_0 = 80$, $k = 0.1$ (condone $k = -0.1$) and their h to find a numerical expression for $\theta_1 = ...$
A1:	Finds the approximation for θ at 1.5 minutes as 71
M1:	Uses their 71 and $k = 0.1$ (condone $k = -0.1$) to find $\frac{d\theta}{dt}$
A1:	Applies the approximation formula again to give $63.35\,(°\text{C})$ or awrt $63\,(°\text{C})$
Note:	$h = 0.1 \Rightarrow \theta_1 = 79.4, \theta_2 = 78.806$; $h = 1 \Rightarrow \theta_1 = 74, \theta_2 = 68.6$; $h = 0.15 \Rightarrow \theta_1 = 79.1, \theta_2 = 78.2135$
(b)	
B1:	See scheme
Note:	Allow B1 for "the value of k should satisfy $0 < k < 0.1$"
Note:	Condone "the value of k would need to be decreased" for B1
Note:	Give B0 for "change k to become negative"

Q2.

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td></td><td>Step length = 0.4</td><td>B1</td><td>1.1b</td></tr>
<tr><td></td><td>
<table>
<tr><td></td><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td></tr>
<tr><td>x</td><td>0.4</td><td>0.8</td><td>1.2</td><td>1.6</td><td>2</td></tr>
<tr><td>y</td><td>$e^{0.16}$</td><td>$e^{0.64}$</td><td>$e^{1.44}$</td><td>$e^{2.56}$</td><td>e^{4}</td></tr>
<tr><td></td><td>1.173…</td><td>1.896…</td><td>4.220…</td><td>12.935…</td><td>54.598…</td></tr>
</table>
</td><td>M1</td><td>1.1b</td></tr>
<tr><td></td><td>$y_0+4y_1+2y_2+4y_3+y_4=123.54....$</td><td>M1</td><td>1.1b</td></tr>
<tr><td></td><td>$\int_{0.4}^{2} e^{x^2}\,dx \approx \frac{0.4}{3}\times\{1.173...+54.598...+4(1.896...+12.935...)+2(4.220...)\}$
$\approx \frac{0.4}{3}\times\text{"123.54..."}$</td><td>dM1</td><td>1.1b</td></tr>
<tr><td></td><td>$=16.5$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>(5)</td><td></td></tr>
<tr><td colspan="4" align="right">(5 marks)</td></tr>
<tr><td colspan="4" align="center">Notes</td></tr>
</table>

B1: Correct step length of 0.4 which may be implied e.g. by their 0.4, 0.8, etc.

M1: Attempts to find y values for their x values – may be in terms of e or numerical values. Must see an attempt to find at least 3 values.

M1: Correct structure for y values of Simpson's rule (ends + 2evens + 4odds) (must have an odd number of ordinates). Must be y values **not** x values.

dM1: $\frac{\text{"0.4"}}{3}\times\text{their}123.54...$ or for $\frac{h}{3}\times\text{their}123.54...$ leading to a value and where h has clearly been defined earlier.

Dependent on both previous method marks

A1: Awrt 16.5

Note that a minimum we would expect to see for full marks is:

$h=0.4$

	y_0	y_1	y_2	y_3	y_4
x	0.4	0.8	1.2	1.6	2
y	$e^{0.16}$	$e^{0.64}$	$e^{1.44}$	$e^{2.56}$	e^{4}
	1.173…	1.896…	4.220…	12.935…	54.598…

$$A\approx\frac{h}{3}[y_0+4y_1+2y_2+4y_3+y_4]=16.5$$

(Note that a calculator gives 16.030…for the area)

Q3.

Question	Scheme	Marks	AOs
	$t_0 = \frac{1}{2}$ and steps are 2 months, so $h = \frac{1}{6}$ $\left(t_1 = \frac{2}{3}, t_2 = \frac{5}{6}\right)$	B1	3.3
	$\left(\frac{dP}{dt}\right)_0 = \frac{540}{5000}\left(1000 - \frac{540\times\left(\frac{1}{2}+1\right)}{6\times\frac{1}{2}+5}\right) = ...\left(97.065 = \frac{19413}{200}\right)$	M1	3.4
	So when $t = \frac{2}{3}$, $P_1 = 540 + \frac{'1'}{6}\times'97.065' = ...$ Or starts with $'97.065' = \frac{y_1 - 540}{1/6}$ and rearranges to find $P_1 = ...$	M1	1.1b
	$= \frac{222471}{400} = 556.1775$	A1	1.1b
	$\left(\frac{dP}{dt}\right)_1 = \frac{'556.1775'}{5000}\left(1000 - \frac{'556.1775'\times\left(\frac{2}{3}+1\right)}{6\times\frac{2}{3}+5}\right) = ...(99.778...)$	M1	3.4
	So when $t = \frac{5}{6}$, $P_2 = '556.1775' + \frac{1}{6}\times'99.778...' = ...(572.807...)$	M1	1.1b
	So there are estimated to be 572 or 573 deer after 10 months.	A1	3.2a
		(7)	
			(7 marks)

Notes:

B1: Uses the given information to set up correct parameters for the model, $t_0 = \frac{1}{2}$, $\left(t_1 = \frac{2}{3}, t_2 = \frac{5}{6}\right)$ and $h = \frac{1}{6}$ seen or implied.

M1: Uses $P_0 = 540$ and their value for t_0 in the given equation to find a value for $\left(\frac{dP}{dt}\right)_0$

M1: Applies the approximation formula with 540, their h and their $\left(\frac{dP}{dt}\right)_0$ to find a value for P_1

A1: Correct approximation P at $t = \frac{2}{3}$. Accept awrt 556.2

M1: Uses $t_1 = t_0 + h$ and their P_1 in the given equation to find a value for $\left(\frac{dP}{dt}\right)_1$.

M1: Uses the approximation a second time with their h, their P_1 and their $\left(\frac{dP}{dt}\right)_1$ to find a value for P_2

A1: Correct answer. Accept either 572 or 573.

Useful table of values for reference

n	P_n	t	$\frac{dP}{dt}$	$h\frac{dP}{dt}$
0	540	$\frac{1}{2}$	97.065	16.1775
1	556.1775	$\frac{2}{3}$	99.77870698	16.6297845
2	572.807	$\frac{5}{6}$		

Note use of $t_0 = 6$ and $h = 2$ leads to $\left(\frac{dP}{dt}\right)_0 = \frac{100494}{1025} = 98.04...$ $P_1 = 736.08...$ $\left(\frac{dP}{dt}\right)_1 = 128.8...$

$P_2 = 993.7...$ which scores a maximum of B0 M1 M1 A0 M1 M1 A0

Q4.

Question	Scheme	Marks	AOs
	{The population after 4 months is required over two iterations} $\Rightarrow h = \frac{1}{6}$	B1	3.3
	$\{t_0 = 0, R_0 = 20 \Rightarrow\}\ \left(\frac{dR}{dt}\right)_0 = 2(20) + 4\sin 0\ \{= 40\}$	M1	3.4
	$\left\{\frac{R_1 - 20}{"(\frac{1}{6})"} = "40" \Rightarrow\right\}\ R_1 = 20 + "\tfrac{1}{6}""(40)"$	M1	1.1b
	$R_1 = \frac{80}{3}$ or awrt 26.7 or $20 + (\text{their } h)(40)$	A1ft	1.1b
	$\left(\frac{dR}{dt}\right)_1 = 2("R_1") + 4\sin("h") = 2\left(\frac{80}{3}\right) + 4\sin\left(\frac{1}{6}\right)\ \{= 53.9969...\}$	M1	1.1b
	$R_2 = R_1 + h\left(\frac{dR}{dt}\right)_1 = \frac{80}{3} + \frac{1}{6}(53.9969...) = 35.666... = 35$ or 36 rabbits	A1	1.1b
	$R_2 = 35.666... \approx 35$ or $36 < 40$ Julie will not be able to start to sell her rabbits after 4 months.	B1ft	3.2a
		(7)	
			(7 marks)

Notes for Question	
B1:	Translates the situation given to state (or use) the correct value for the step length h
M1:	Uses the model to find the initial value of $\frac{dR}{dt}$ using the initial condition $t_0 = 0, R_0 = 20$
M1:	Applies the approximation formula with $R_0 = 20$, their stated h, their $\left(\frac{dR}{dt}\right)_0$ to find a numerical expression for R_1
A1:	**depends on both previous M marks** At 2 months, finds the approximation for R as $\frac{80}{3}$ or awrt 26.7
Note:	Only give the following follow through. i.e. Allow A1ft for $20 + (\text{their } h)(40)$ for their stated h
M1:	Attempts to find a numerical expression for $\left(\frac{dR}{dt}\right)_1$ with their $\frac{80}{3}$ and $t_1 =$ their h
A1:	Applies the approximation formula for a second time to give R_2 as a truncated 35 or a value in the interval [35.5, 36]
B1ft:	Attempts two iterations of their $R_{n+1} = R_n + h\left(\frac{dR}{dt}\right)_n$ to find a value for R_2. Compares their value of R_2 with 40 (which can be implied) and draws a conclusion about whether Julie will be able to start to sell her rabbits after 4 months.
Note:	Give final B0 for applying more than or fewer than two iterations before comparing
Note:	Using $h = \frac{1}{12}$ yields $R_1 = 23.3333..., R_2 = 27.2499..., R_3 = 31.8469..., R_4 = 37.2372...$
Note:	Give special case final A1 for giving R_4 as a truncated 37 or a value in the interval [37, 37.4]
Note:	Therefore, using $h = \frac{1}{12}$ with four iterations can gain a maximum B0 M1 M1 A1 M1 A1 B0
Note:	Answers in the range [35.5, 36] can follow from an incorrect method. E.g. Give final M0 A0 for using $h = \frac{1}{6}$, $\left(\frac{dR}{dt}\right)_1 = 2\left(\frac{80}{3}\right) + 4\sin(\underline{0.1}) = 53.73266... \Rightarrow R_2 = \frac{80}{3} + \frac{1}{6}(53.73266...) = 35.622...$

Q5.

Question	Scheme	Marks	AOs
	$\frac{d^2y}{dx^2}=2y^2-x-1\Rightarrow\left(\frac{d^2y}{dx^2}\right)_0=0-0-1=-1$	B1	1.1b
	$\left(\frac{dy}{dx}\right)_0=3\Rightarrow\frac{(y_1-y_{-1})}{0.2}\approx 3$	B1	1.1b
	$\left(\frac{d^2y}{dx^2}\right)_0\approx\frac{(y_1-2y_0+y_{-1})}{h^2}\Rightarrow\frac{y_1-2(0)+y_{-1}}{0.01}\approx -1$	M1	1.1b
	$y_1\approx\frac{1}{2}(0.6-0.01)=0.295$	dM1	2.1
	$\frac{d^2y}{dx^2}=2y^2-x-1\Rightarrow\left(\frac{d^2y}{dx^2}\right)_1=2(0.295)^2-0.1-1=-0.92595$	dM1	1.1b
	$\left(\frac{d^2y}{dx^2}\right)_1\approx\frac{(y_2-2y_1+y_0)}{h^2}\Rightarrow\frac{y_2-2(0.295)+0}{0.01}\approx -0.92595\Rightarrow y_2=...$	dM1	2.1
	$y_2\approx 2(0.295)-0.92595\times 0.01=0.581$ (3 s.f.)	A1	1.1b
		(7)	
			(7 marks)

Notes

B1: Correct value for the second derivative using the differential equation
B1: Correct equation in terms of y_1 and y_{-1} using the first order approximation
M1: Uses the second order approximation to obtain another equation in terms of y_1 and y_{-1}
M1: Uses their two equations in y_1 and y_{-1} and solves together to find y at $x = 0.1$.
M1: Uses the differential equation with their y at $x = 0.1$ and $x = 0.1$ to find a value for the second derivative at $x = 0.1$
M1: Completes the process by using the second order approximation and their second derivative to obtain a value for y_2
A1: Correct value for y at $x = 0.2$
Note that all method marks are dependent

Q6.

Question	Scheme	Marks	AOs
(a)	At 6 hours $t = 0.25$ so "h" is 0.25	B1	3.1b
	At $t = 0$ $\frac{dx}{dt} = \frac{3+\cosh 0}{3\times 3^2 \cosh 0} - \frac{1}{3}(3)\tanh 0 = ...\left(=\frac{4}{27}\right)$	M1	3.4
	So $x_1 \approx 3 + "0.25" \times "\frac{4}{27}" = ...$	M1	1.1b
	After 6 hours concentration of the pollutant is approximately awrt 3.04 ppm (3 s.f.) or $\frac{82}{27}$ ppm	A1	3.2a
		(4)	
(b)	$\frac{du}{dt} = 3x^2 \times \frac{dx}{dt}$ or $\frac{1}{3x^2}\frac{du}{dt} = \frac{dx}{dt}$ or $\frac{du}{dt} = \frac{du}{dx}\times\frac{dx}{dt} = 3x^2\left(\frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t\right)$	B1	2.2a
	So $\frac{dx}{dt} = \frac{3+\cosh t}{3x^2 \cosh t} - \frac{1}{3}x\tanh t \rightarrow \frac{1}{3x^2}\frac{du}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t$ $\rightarrow \frac{du}{dt} = \frac{3}{\cosh t} + 1 - u\tanh t$ $\frac{dx}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \rightarrow \frac{1}{3u^{\frac{2}{3}}}\frac{du}{dt} = \frac{3+\cosh t}{3u^{\frac{2}{3}}\cosh t} - \frac{1}{3}u^{\frac{1}{3}}\tanh t$ $\rightarrow \frac{du}{dt} = \frac{3}{\cosh t} + 1 - u\tanh t$	M1	2.1
	$\frac{du}{dt} + u\tanh t = 1 + \frac{3}{\cosh t}$ *	A1*	1.1b
		(3)	
(c)	I.F. $= \exp\left(\int \tanh t \, dt\right) = \exp(\ln\cosh t) = \cosh t$	B1	2.2a
	$\Rightarrow u'\cosh t' = \int '\cosh t'\left(1 + \frac{3}{\cosh t}\right) dt = \left\{\int \cosh t + 3 \, dt\right\}$	M1	1.1b
	$\Rightarrow u\cosh t = \sinh t + 3t (+c)$	M1	1.1b
	$u\cosh t = \sinh t + 3t + c$ or $u = \tanh t + \frac{3t}{\cosh t} + \frac{c}{\cosh t}$ oe	A1	1.1b
		(4)	
(d)	$t = 0 \Rightarrow x = 3, u = 27 \Rightarrow c = 27\cosh 0 - \sinh 0 - 3(0) = 27$	M1	3.4
	$\Rightarrow x = \left(\tanh t + \frac{3t + "27"}{\cosh t}\right)^{\frac{1}{3}}$	M1	3.4

	$x=\left(\tanh t+\dfrac{3t+27}{\cosh t}\right)^{\frac{1}{3}}$ (oe)	A1	3.2a
		(3)	
(e)	$x(0.25)=\left(\tanh 0.25+\dfrac{(0.75+27)}{\cosh 0.25}\right)^{\frac{1}{3}}=...(=3.0055...)$	M1	3.4
	% error is $\dfrac{3.0055...-3.037...}{3.0055...}\times 100=...$	M1	1.1b
	Estimate in (a) is an overestimate by 1.05% (3 s.f.)	A1	3.2a
		(3)	
			(17 marks)

Notes:
(a) **B1:** Identifies a correct step length for the situation – 6 hours is a quarter of a day, so $h = 0.25$ **M1:** Uses $"y_0"=x(0)=3$ and $t=0$ to find $"\left(\dfrac{dy}{dx}\right)_0"=\left(\dfrac{dx}{dt}\right)_0$. Accept with whichever notation used, as long as it is clear they are attempting the correct things. **M1:** Applies the approximation formula with their "h" and their $"\left(\dfrac{dy}{dx}\right)_0"$ **A1:** For awrt 3.04 ppm. Accept $\dfrac{82}{27}$ ppm
(b) **B1:** A correct equation relating $\dfrac{du}{dt}$ and $\dfrac{dx}{dt}$ from the chain rule. **M1:** Makes a complete substitution for x and $\dfrac{dx}{dt}$ in equation (I) or a complete substitution for u and $\dfrac{du}{dt}$ in equation (II) **A1***: Simplifies correctly to achieve the given result.
(c) **B1:** Correct integrating factor found or spotted. Allow for $e^{\ln\cosh t}$ **M1:** Applies IF to achieve $u"\cosh t"=\int "\cosh t"\left(1+\dfrac{3}{\cosh t}\right)dt$ **M1:** A reasonable attempt to integrate the RHS. Need not include constant of integration. If I.F. correct allow for $\pm\sinh t+3t(+c)$ **A1:** Correct general solution, either implicit or explicit form including the context of integration (award when first seen and isw)
(d) **M1:** Uses the initial conditions in an appropriate equation to find the constant of integration. Either $t = 0$ and $u = 27$ in the answer to (c), or $t = 0$ and $x = 3$ if substitution for x occurs first.

M1: Reverses the substitution and rearranges to find equation for x, with evaluated constant included. **A1:** Correct equation, any equivalent form, but must be $x = \dots$
(e) **M1:** Uses their model solution to find the value at $t = 0.25$ **M1:** Applies $\frac{\text{actual value} - \text{estimate}}{\text{actual value}} \times 100$ with their values. **A1:** States part (a) is overestimate by 1.05%

Q7.

Question	Scheme	Marks	AOs
	$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 \approx \frac{(y_2 - 1)}{0.2}$	B1	1.1b
	$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1 \approx \frac{(y_2 - 2(2) + 1)}{0.1^2}$	B1	1.1b
	$\frac{(y_2 - 2(2)+1)}{0.1^2} + 15\left(\frac{(y_2 - 1)}{0.2}\right) - 3(2)^2 = 2(0.1) \Rightarrow y_2 = \dots$	M1	2.1
	$y_2 \approx \frac{1936}{875}$ (2.2125...)	A1	1.1b
	$\frac{\left(y_3 - 2\left(\frac{1936}{875}\right) + 2\right)}{0.1^2} + 15\left(\frac{y_3 - 2}{0.2}\right) - 3\left(\frac{1936}{875}\right)^2 = 2(0.2) \Rightarrow y_3 = \dots$	M1	2.1
	$y_3 \approx 2.32914\dots$	A1	1.1b
		(6)	
			(6 marks)

Notes

B1: Correct expression for the first derivative using the given values and the approximation
B1: Correct expression for the second derivative using the given values and the approximation
M1: Uses the approximations for the first and second derivatives, substitutes into the differential equation and obtains a value for y at $x = 0.2$
A1: Correct value for y at $x = 0.2$ (accept the exact value or awrt 2.21)
M1: Completes the process by using their value for y at $x = 0.2$ to obtain a value for y at $x = 0.3$
A1: Correct value for y when $x = 0.3$ (allow awrt 2.33)

Q8.

Question	Scheme	Marks	AOs
(a)	Identifies $t_0 = 0,\ v_0 = 0,\ \left(\frac{dv}{dt}\right)_0 = 10$ and $h = 0.5$	B1	3.4
	$v_1 = v_0 + h\left(\frac{dv}{dt}\right)_0 \Rightarrow v_1 = 0 + 0.5 \times 10 = \ldots$	M1	1.1b
	$v_1 = 5$	A1	1.1b
	$\left(\frac{dv}{dt}\right)_1 = -0.1(5)^2 + 10 = \ldots\{7.5\}$ $v_2 = v_1 + h\left(\frac{dv}{dt}\right)_1 \Rightarrow v_2 = 5 + 0.5 \times 7.5 = \ldots$	M1	3.4
	$v_2 = 8.75$ so 8.75 ms^{-1}	A1	1.1b
		(5)	
(b)	$\frac{dv}{dt} = -0.1v^2 + A$ where $0 < A < 10$	B1	3.5c
		(1)	
			(6 marks)

Notes:

(a)

B1: Uses the model to identify the correct initial conditions and requirements for h. May be implied by use in the equation.

M1: Applies the approximation formula with their values for v_0, $\left(\frac{dv}{dt}\right)_0$ and h to find a value for v_1

A1: $v_1 = 5$

M1: Uses their v_1 to find a value for $\left(\frac{dv}{dt}\right)_1$ and applies the approximation formula with their values for v_1, $\left(\frac{dv}{dt}\right)_1$ and h to find a value for v_2

A1: $v_2 = 8.75$or 8.75 ms^{-1}

(b)

B1: Reduce the value of 10 or explains this is what needs **reducing**, but do not accept 0 or negative values in place of the 10. Note: "change the 10" is B0 if it does not explain how to change it.

Q9.

Question	Scheme	Marks	AOs
	Step $\frac{1}{3}$	B1	1.1b
	<table><tr><td></td><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td><td>y_5</td><td>y_6</td></tr><tr><td>x</td><td>-1</td><td>$-\frac{2}{3}$</td><td>$-\frac{1}{3}$</td><td>0</td><td>$\frac{1}{3}$</td><td>$\frac{2}{3}$</td><td>1</td></tr><tr><td>y</td><td>0</td><td>2.2981</td><td>2.9544</td><td>3</td><td>2.9544</td><td>2.2981</td><td>0</td></tr></table>	M1	3.4
	$y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6=\text{"42.203"}$ $\{0+4(2.2981+3+2.2981)+2(2.9544+2.9544)+0\}$	M1	1.1b
	$=42.203\left(=24\cos\left(\frac{2\pi}{9}\right)+12\cos\left(\frac{\pi}{18}\right)+12\right)$	A1	1.1b
	So volume required is approx. $\frac{85}{1000}\times\frac{\frac{1}{3}}{3}\times\text{"42.203"}$	M1	3.1a
	= awrt 0.3986 m^3	A1	3.2a
	Alternative interval [0,1]) step $\frac{1}{6}$ and the answer is doubled later	B1	1.1b
	<table><tr><td></td><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td><td>y_5</td><td>y_6</td></tr><tr><td>x</td><td>0</td><td>$\frac{1}{6}$</td><td>$\frac{1}{3}$</td><td>$\frac{1}{2}$</td><td>$\frac{2}{3}$</td><td>$\frac{5}{6}$</td><td>1</td></tr><tr><td>y</td><td>3</td><td>2.9971</td><td>2.9544</td><td>2.7716</td><td>2.2981</td><td>1.3852</td><td>0</td></tr></table>	M1	3.4
	$y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6=\text{"42.1206"}$ $\{3+4(2.9971+2.7716+1.3852)+2(2.9544+2.2981)+0\}$	M1	1.1b
	Awrt 42.121	A1	1.1b
	So volume required is approx. $\frac{85}{1000}\times\frac{\frac{1}{6}}{3}\times\text{"42.1206"}\times 2$	M1	3.1a
	= awrt 0.3978 m^3	A1	3.2a
		(6)	
		(6 marks)	

Notes:

B1: Correct strip width for the method chosen $\frac{1}{3}$ for the interval $[-1,1]$

M1: Uses the model to find the appropriate values for the method. May use that the function is even to only work out half of them, so may be implied by use in the formula. At least two correct values to 4 s.f. needed for the method.

M1: Applies the "bracket" of Simpson's rule, "$y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6$". Coefficients must be correct.

A1: Correct value for the "bracket". If not explicitly seen, may be implied by awrt 4.689 as a value for the cross section area following correct values.

M1: Correct full method to find the volume. E.g. multiplies their bracket by their $\frac{h}{3}$ and by 0.085.

Accept an attempt in any consistent units, so e.g. in mm^3 ie $85\times\frac{\frac{1}{3}}{3}\times\text{"42.203"}\times 1000^2$

A1: Correct answer in m^3.

B1: Correct strip width for the method chosen $\frac{1}{6}$ for the interval [0,1] and later doubled.

M1: Uses the model to find the appropriate values for the method. May use that the function is even to only work out half of them, so may be implied by use in the formula. At least two correct values to 4 s.f. needed for the method.

M1: Applies the "bracket" of Simpson's rule, $"y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6"$. Coefficients must be correct.

A1: Correct value for the "bracket". If not explicitly seen, may be implied by awrt 4.680 as a value for the cross section area following correct values.

M1: Correct full method to find the volume. E.g. multiplies their bracket by their $\frac{h}{3}$ and by 0.085.

Accept an attempt in any consistent units, so e.g. in mm^3 ie $85\times\frac{\frac{1}{6}}{3}\times"42.203"\times1000^2\times2$

A1: Correct answer in m^3.

__**Using 6 ordinates**__

Max score B0 M1 M0 A0 M0 A0

	y_0	y_1	y_2	y_3	y_4	y_5
x	-1	-0.6	-0.2	0.2	0.6	1
y	0	2.53298	2.9941	2.9941	2.53298	0

B0: Incorrect strip width

M1: Uses the model to find the appropriate values for the method. At least two correct values to 4 s.f. needed for the method.

Q10.

Question	Scheme	Marks	AOs
	£300 purchased one hour after opening $\Rightarrow V_0 = 3$ and $t_0 = 1$; half an hour after purchase $\Rightarrow t_2 = 1.5$, so step h required is 0.25	B1	3.3
	$t_0 = 1,\ V_0 = 3,\ \left(\frac{dV}{dt}\right)_0 \approx \frac{3^2 - 1}{1^2 + 3} = 2$	M1	3.4
	$V_1 \approx V_0 + h\left(\frac{dV}{dt}\right)_0 = 3 + 0.25 \times 2 = \ldots$	M1	1.1b
	$= 3.5$	A1ft	1.1b
	$\left(\frac{dV}{dt}\right)_1 \approx \frac{3.5^2 - 1.25}{1.25^2 + 1.25 \times 3.5} \left(= \frac{176}{95}\right)$	M1	1.1b
	$V_2 \approx V_1 + h\left(\frac{dV}{dt}\right)_1 = 3.5 + 0.25 \times \frac{176}{95} = 3.963\ldots$, so £396 (nearest £)	A1	3.2a
		(6)	
			(6 marks)

Notes

B1	Identifies the correct initial conditions and requirement for h.
M1	Uses the model to evaluate $\frac{dV}{dt}$ at t_0, using their t_0 and V_0.
M1	Applies the approximation formula with their values.
A1ft	3.5 or exact equivalent. Follow through their step value.
M1	Attempt to find $\left(\frac{dV}{dt}\right)_1$ with their 3.5
A1	Applies the approximation and interprets the result to give £396.

Q11.

Question	Scheme	Marks	AOs
	Step 0.5	B1	1.1b
	<table><tr><td></td><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td><td>y_5</td><td>y_6</td></tr><tr><td>x</td><td>1</td><td>1.5</td><td>2</td><td>2.5</td><td>3</td><td>3.5</td><td>4</td></tr><tr><td>y</td><td>$\sqrt{2}$</td><td>$\sqrt{4.375}$</td><td>3</td><td>$\sqrt{16.625}$</td><td>$\sqrt{28}$</td><td>$\sqrt{43.875}$</td><td>$\sqrt{65}$</td></tr></table>	M1	1.1b
	$y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6 = "77.23"$	M1	1.1b
	$\int_1^4 \sqrt{1+x^3}\,dx \approx \frac{0.5}{3} \times "77.23"$	M1	1.1b
	$= 12.9$	A1	1.1b
		(5)	
		(5 marks)	

Notes

B1 Use of step length 0.5

M1 Attempt to find y values with at least 2 correct.

M1 Use of formula "$y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6$" with correct coefficients

A1 $\frac{0.5}{3} \times$ their 77.23

A1 awrt 12.9