Methods in Calculus (FP1)

Questions

Q1.

$$I = \int \frac{1}{4\cos x - 3\sin x} \mathrm{d}x \qquad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k$$

where k is an arbitrary constant.

(Total for question = 8 marks)

Q2.

Use l'Hospital's Rule to show that

$$\lim_{x \to \frac{\pi}{2}} \frac{(\mathrm{e}^{\sin x} - \cos(3x) - \mathrm{e})}{\tan(2x)} = -\frac{3}{2}$$

(Total for question = 5 marks)

Q3.

Given that *k* is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5 y}{dx^5} = (k^2x^2 + A)k^3 x \cos kx + B(k^2x^2 + C)k^2 \sin kx$$

where *A*, *B* and *C* are integers to be determined.

(Total for question = 4 marks)

Q4.

Given *k* is a constant and that

$$y = x^3 e^{kx}$$

use Leibnitz theorem to show that

$$\frac{d^n y}{dx^n} = k^{n-3} e^{kx} \left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$$

(4)

(Total for question = 4 marks)

Q5.

(i) Use the substitution $t = \tan\frac{x}{2}$ to prove the identity

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} \equiv \sec x + \tan x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(5)

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{5}{4 + 2\cos\theta}\, d\theta$$

giving your answer in simplest form.

(5)

(Total for question = 10 marks)

Q6.

During 2029, the number of hours of daylight per day in London, *H*, is modelled by the equation

$$H = 0.3\sin\left(\frac{x}{60}\right) - 4\cos\left(\frac{x}{60}\right) + 11.5 \qquad 0 \leq x < 365$$

where *x* is the number of days after 1st January 2029 and the angle is in radians.

(a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures.

(1)

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that *H* can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2}$$

where *a*, *b* and *c* are constants to be determined.

(2)

(c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London.

(4)

(Total for question = 7 marks)

Mark Scheme – Methods in Calculus (FP1)

Q1.

Question	Scheme	Marks	AOs
	$4\cos x - 3\sin x = 4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$	B1	1.1a
	$\frac{dt}{dx} = \frac{1+t^2}{2}$ or $\frac{dx}{dt} = \frac{2}{1+t^2}$ or $dx = \frac{2dt}{1+t^2}$ or $dt = \frac{1+t^2}{2}dx$ oe	B1 **M1 on ePEN**	2.1
	$\int \frac{1}{4\cos x - 3\sin x}dx = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$	M1	2.1
	$= \int \frac{2}{4-4t^2-6t}(dt)$ or $\int \frac{1}{2-2t^2-3t}(dt)$ or $\int \frac{-1}{2t^2+3t-2}(dt)$ etc.	A1	1.1b
	$\frac{-2}{4t^2+6t-4} = \frac{-1}{(t+2)(2t-1)} = \frac{A}{(t+2)} + \frac{B}{(2t-1)}$ $\frac{-1}{(t+2)(2t-1)} = \frac{1}{5(t+2)} + \frac{2}{5(1-2t)}$	M1	3.1a
	$\Rightarrow I = \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}(dt)$ or equivalent	A1	1.1b
	$= \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}dt = \frac{1}{5}\ln(t+2) - \frac{1}{5}\ln(1-2t)(+k)$	A1	1.1b
	$= \frac{1}{5}\ln\left(\frac{2+t}{1-2t}\right)(+k) = \frac{1}{5}\ln\left(\frac{2+\tan\left(\frac{x}{2}\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$*	A1*	2.1
		(8)	
	Alternative for final 4 marks:		
	$= \int \frac{2}{4-4t^2-6t}(dt) = -\frac{1}{2}\int \frac{1}{t^2+\frac{3}{2}t-1}(dt) = -\frac{1}{2}\int \frac{1}{\left(t+\frac{3}{4}\right)^2 - \frac{25}{16}}(dt)$ or e.g. $\int \frac{1}{\frac{25}{8} - 2\left(t+\frac{3}{4}\right)^2}(dt)$	M1 A1	3.1a 1.1b
	$-\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5}\ln\left(\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}}\right)(+c)$	A1	1.1b
	$-\frac{1}{5}\ln\left\lvert\frac{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}{\tan\left(\frac{x}{2}\right)+2}\right\rvert + c = \frac{1}{5}\ln\left\lvert\frac{\tan\left(\frac{x}{2}\right)+2}{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}\right\rvert + c = \frac{1}{5}\ln\left(\frac{\tan\left(\frac{x}{2}\right)+2}{\frac{1}{2}-\tan\left(\frac{x}{2}\right)}\right) + c$ $= \frac{1}{5}\ln\left(\frac{2\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + c = \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + \frac{1}{5}\ln 2 + c$ $= \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$	A1*	2.1
		(8 marks)	

Notes
B1: Uses the **correct** formulae to express $4\cos x - 3\sin x$ in terms of t B1(M1 on ePEN): Correct equation in terms of dx, dt and t – can be implied if seen as part of their substitution. M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of "dx" but must be dx = f(t)dt where f(t) ≠ 1. This mark is also available if the candidate makes errors when attempting to simplify $4\left(\frac{1-t^2}{1+t^2}\right)-3\left(\frac{2t}{1+t^2}\right)$ before attempting the substitution. A1: For obtaining a fully correct simplified integral with a constant in the numerator and a 3 term quadratic expression in the denominator. ("dt" not required) M1: Realises the need to express the integrand in terms of partial fractions in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral in terms of partial fractions – allow any equivalent **correct** integral. ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary. The denominator must also be dealt with correctly. E.g. if it appears as $2t - 1$ initially and becomes $1 - 2t$ without justification, this final mark should be withheld. **<u>Alternative for final 4 marks:</u>** M1: Realises the need to express the integrand in completed square form in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral with the square completed – allow any equivalent **correct** integral ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary as shown in the scheme and with the denominator dealt with correctly if necessary. Note that it is acceptable for the "dt" to appear and disappear throughout the proof as long as the intention is clear.

Q2.

Question	Scheme	Marks	AOs
	$\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}-\cos(3x)-\mathrm{e}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(\tan(2x))}=\dfrac{\pm\cos(x)\mathrm{e}^{\sin x}\pm A\sin(3x)}{B\sec^2 2x}$	M1	1.1b
	$\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}-\cos(3x)-\mathrm{e}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(\tan(2x))}=\dfrac{\cos(x)\mathrm{e}^{\sin x}+3\sin(3x)}{2\sec^2 2x}$	A1 A1	1.1b 1.1b
	$\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos(x)\mathrm{e}^{\sin x}+3\sin(3x)}{2\sec^2 2x}$ $=\dfrac{\cos\left(\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(\frac{\pi}{2}\right)}+3\sin\left(\frac{3\pi}{2}\right)}{2\sec^2\left(\frac{2\pi}{2}\right)}$ or $=\dfrac{0\times\mathrm{e}+3\times(-1)}{2\times(-1)^2}=\ldots$	M1	1.2
	$=-\dfrac{3}{2}$ *	A1*	2.1
		(5)	
			(5 marks)

Notes:

M1: Attempts differentiation of both numerator and denominator, including at least one use of the chain rule. Either numerator or denominator of the correct form. May be done separately.

A1: Numerator correct

A1: Denominator correct

M1: Applies l'Hospital's Rule, must see **clear use of a substitution** of $x=\frac{p}{2}$ into their derivatives, not the original expression. (no need to see check that limits of numerator and denominator are non-zero).

A1*: Needs to be a correct intermediate line following substitution before reaching the printed answer with use of some limit notation. All aspects of the proof should be clear for this mark to be awarded and no errors seen.

Q3.

Question	Scheme	Marks	AOs
	$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2, \frac{d^2u}{dx^2} = 6x, \frac{d^3u}{dx^3} = 6$	M1	1.1b
	$v = \sin kx \Rightarrow \frac{dv}{dx} = k\cos kx, \frac{d^2v}{dx^2} = -k^2 \sin kx, \frac{d^3v}{dx^3} = -k^3 \cos kx,$ $\frac{d^4v}{dx^4} = k^4 \sin kx, \frac{d^5v}{dx^5} = k^5 \cos kx$	M1	2.1
	$\frac{d^5y}{dx^5} = x^3k^5 \cos kx + 5\times 3x^2 \times k^4 \sin kx + \frac{5\times 4}{2}\times 6x \times \left(-k^3 \cos kx\right) +$ $\frac{5\times 4\times 3}{3!}\times 6 \times \left(-k^2 \sin kx\right)$	M1	2.1
	$= \left(k^2x^2 - 60\right)k^3 x \cos kx + 15\left(k^2x^2 - 4\right)k^2 \sin kx$	A1	1.1b
		(4)	
			(4 marks)

Notes

M1: Differentiates $u = x^3$ three times. Need to see $x^3 \rightarrow ...x^2 \rightarrow ..x \rightarrow k$

M1: Uses $v = \sin kx$ to establish the form of the derivatives. Need to see at least alternating $k^{...}\sin kx$ and $k^{...}\cos kx$ with increasing powers of k for at least 3 derivatives.

M1: Uses a correct formula with 2 and 3! (or 6) with terms shown to disappear after the fourth term. This needs to be a correct application of the theorem so that the correct binomial coefficients need to go with the correct pairings of their derivatives. If there is any doubt, at least 3 terms should have the correct structure. Allow equivalent notation for the binomial coefficients e.g. $\begin{pmatrix}5\\0\end{pmatrix}, \begin{pmatrix}5\\1\end{pmatrix}$ etc. or ${}^5C_0, {}^5C_1$ etc.

A1: Correct expression in the required form with correct values of A, B and C. Apply isw if necessary e.g. if a correct expression is followed by $A = 60, B = 15, C = -4$
(NB $A = -60, B = 15, C = -4$)

If there is no use of Lebnitz's theorem e.g. repeated differentiation of products, this scores no marks.

Q4.

Question	Scheme	Marks	AOs
	$y = x^3 e^{kx}$ so $u = x^3$ and $\frac{du}{dx} = 3x^2$ and $\frac{d^2u}{dx^2} = 6x$ and $\frac{d^3u}{dx^3} = 6$ $\left(\text{and } \frac{d^4u}{dx^4} = 0\right)$	M1	1.1b
	$v = e^{kx}$ and $\frac{d^n v}{dx^n} = k^n e^{kx}$ and $\frac{d^{n-1}v}{dx^{n-1}} = k^{n-1}e^{kx}$ and $\frac{d^{n-2}v}{dx^{n-2}} = k^{n-2}e^{kx}$ (and...)	M1	2.1
	$\frac{d^n y}{dx^n} = x^3 k^n e^{kx} + n3x^2 k^{n-1} e^{kx} + \frac{n(n-1)}{2} 6xk^{n-2}e^{kx} + \frac{n(n-1)(n-2)}{3!} 6k^{n-3}e^{kx}$ and remaining terms disappear.	M1	2.1
	So $\frac{d^n y}{dx^n} = k^{n-3}e^{kx}\left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$ *	A1*	1.1b
		(4)	
		(4 marks)	

Notes

M1	Differentiate $u = x^3$ three times.
M1	Use $u = e^{kx}$ and establish the form of the derivatives, with at least the three shown.
M1	Uses correct formula, with 2 and 3! (or 6) and with terms shown to disappear after the fourth term.
A1*	Correct solution leading to the given answer stated. No errors seen.

Q5.

Question	Scheme	Marks	AOs
(i)	$\dfrac{\sin x-\cos x+1}{\sin x+\cos x-1}=\dfrac{\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}+1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1}=\ldots$	M1	1.1b
	$=\dfrac{2t-(1-t^2)+1+t^2}{2t+1-t^2-(1+t^2)}$ or $=\dfrac{2t-1+t^2+1+t^2}{2t+1-t^2-1-t^2)}$ numerator $=\dfrac{2t-(1-t^2)+1+t^2}{1+t^2}$ denominator $= =\dfrac{2t+1-t^2-(1+t^2)}{1+t^2}$ and divides	M1	2.1
	$=\dfrac{2t^2+2t}{2t-2t^2}\left(=\dfrac{1+t}{1-t}\right)$	A1	1.1b
	$=\dfrac{t+1}{1-t}\times\dfrac{1+t}{1+t}=\dfrac{t^2+2t+1}{1-t^2}=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}$ Alt: $\sec x+\tan x=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}=\dfrac{1+2t+t^2}{1-t^2}$	M1	3.1a
	$=\dfrac{1}{\cos x}+\tan x=\sec x+\tan x$ * Alt: $=\dfrac{(t+1)^2}{(1-t)(1+t)}=\dfrac{1+t}{1-t}=LHS$ hence result proved.*	A1*	2.1
		(5)	

(ii)	$\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5}{4+2\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}\,\mathrm{d}t$ **Alternatively** $\dfrac{\mathrm{d}t}{\mathrm{d}q}=\dfrac{1}{2}\sec^2\left(\dfrac{q}{2}\right)=\dfrac{1}{\cos q+1}$ leading to $\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5\cos\theta+5}{4+2\cos\theta}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5\left(\frac{1-t^2}{1+t^2}\right)+5}{4+2\left(\frac{1-t^2}{1+t^2}\right)}\,\mathrm{d}t$	M1	2.1
	$\displaystyle=\int_{(0)}^{(1)}\frac{10}{4(1+t^2)+2(1-t^2)}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5}{3+t^2}\,\mathrm{d}t$ o.e.	A1	1.1b
	$=\left[5\times\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{t}{\sqrt{3}}\right)\right]_{(0)}^{(1)}$	M1	1.1b

	$=\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-0\right)$ or $\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{\tan\left(\frac{\pi}{4}\right)}{\sqrt{3}}\right)-\arctan\left(\frac{\tan(0)}{\sqrt{3}}\right)\right)$	**M1**	2.2a
	$=\frac{5\pi\sqrt{3}}{18}$ oe in a surd form e.g. $\frac{5\pi}{6\sqrt{3}}$	**A1**	1.1b
		(5)	
			(10 marks)

Notes:
(i) **M1:** Applies the t-formulae to the left-hand side of expression. Allow slips in signs of the terms. **M1:** Multiplies numerator and denominator through by $1+t^2$ (allow if they forget to multiply the 1's). Alternative works separately on the numerator and denominator to combine terms and then divides. **A1:** Correct $\frac{\text{quadratic}}{\text{quadratic}}$ with terms gathered, award where first seen, need not have cancelled $2t$ for this mark. **M1:** Cancels $2t$, multiplies numerator and denominator by $1+t$ and splits to sum of two terms. If working from both sides, this mark is for substituting the t-formulae into the right-hand side and combining to single fraction. **A1*:** Correct completion to given result. No errors in proof. If working from both sides, a suitable conclusion is needed, e.g "hence proven".
(ii) **M1:** Applies the substitution including the use of $\mathrm{d}\theta=\frac{2}{1+t^2}\mathrm{d}t$ (Limits not needed for first three marks). **A1:** Simplifies correctly to a recognisable integrable form. **M1:** Integrates to the form $K\arctan\left(\frac{t}{a}\right)$ where a^2 is their constant term. **M1:** Deduces correct limits and applies them the correct way round OR deduces integral in terms of θ from their integration and applies original limits the correct way. **A1:** $\frac{5\pi\sqrt{3}}{18}$ or equivalent in surd form. Note use of calculator does not lead to the exact value required in the question = 1.51149947 This can score M1 A1 M0 M1 A0

Q6.

Question	Scheme	Marks	AOs
(a)	$(H =)0.3\,sin\left(\frac{30}{60}\right) - 4\,cos\left(\frac{30}{60}\right) + 11.5 = 8.13$ {hours}*	B1*	3.4
		(1)	
(b)	Substitutes $sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}$ and $cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$into H $$(H =)0.3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 11.5$$	M1	1.1b
	$(H =)\dfrac{0.6t - 4 + 4t^2 + 11.5(1+t^2)}{1+t^2} = \dfrac{15.5t^2 + 0.6t + 7.5}{1+t^2}$	A1	2.1
		(2)	
(c)	$H = \dfrac{15.5t^2 + 0.6t + 7.5}{1+t^2} = 12 \Rightarrow 3.5t^2 + 0.6t - 4.5 = 0$	M1	3.4
	$\Rightarrow t = \dfrac{-0.6 \pm \sqrt{0.6^2 - 4(3.5)(-4.5)}}{7}$ $= \ldots(1.051\ldots, -1.222\ldots) \Rightarrow x = 120\,tan^{-1}(\text{"}1.051..)$ $= \ldots(97.254..)$	dM1	3.1b
	$x =$ awrt 97	A1	1.1b
	8th or 9th April	A1	3.2a
		(4)	
			(7 marks)

Notes:
(a) **B1***: Uses $x = 30$ to show that $H = 8.13$. Accept $x = 30$ seen substituted followed by 8.13, or $x = 30$ identified followed by 8.133… before rounding to 8.13.
(b) **M1**: Uses the correct t-formulae $sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}andcos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$, attempts to substitute into H. **A1**: Fully correct method, expresses as a single fraction with a denominator of $1 + t^2$ to achieve $H = \frac{15.5t^2+0.6t+7.5}{1+t^2}$ (oe with fractions or accept values for a, b and c stated).
(c) **M1**: Sets $H = 12$ (or any inequality in between) and rearranges to form a a quadratic equation for t. **dM1**: Dependent on the previous method mark. Solves the quadratic by any means (accept one correct answer for their quadratic if no method shown) and uses this to find a value for x. **A1**: Correct value for $x =$ awrt 97 or accept 98 following a correct value for t. **A1**: Correct day of the year. Accept 8th or 9th April following awrt 97 from a correct method. **Note**: Question says hence, so answers by graphical methods or trial and improvement are not acceptable for full credit. They can score a SC M0dM0A0B1 for achieving a correct date.