Linear Transformations

Questions

Q1.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation *U* represented by the matrix **A**.

(3)

The transformation *V*, represented by the 2 × 2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**.

(1)

Given that *U* followed by *V* is the transformation *T*, which is represented by the matrix **C**,

(c) find the matrix **C**.

(2)

(d) Show that there is a real number *k* for which the point (1, *k*) is invariant under *T*.

(4)

(Total for question = 10 marks)

Q2.

(i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$$

where *a* and *b* are non-zero constants.
Given that the matrix **A** is self-inverse,
(a) determine the value of *b* and the possible values for *a*.

(5)

The matrix **A** represents a linear transformation *M*.
Using the smaller value of *a* from part (a),
(b) show that the invariant points of the linear transformation *M* form a line, stating the equation of this line.

(3)

(ii)

$$\mathbf{P} = \begin{pmatrix} p & 2p \\ -1 & 3p \end{pmatrix}$$

where p is a positive constant.
The matrix **P** represents a linear transformation U.
The triangle T has vertices at the points with coordinates (1, 2), (3, 2) and (2, 5).
The area of the image of T under the linear transformation U is 15
(a) Determine the value of p.

(4)

The transformation V consists of a stretch scale factor 3 parallel to the x-axis with the y-axis invariant followed by a stretch scale factor –2 parallel to the y-axis with the x-axis invariant. The transformation V is represented by the matrix **Q**.
(b) Write down the matrix **Q**.

(2)

Given that U followed by V is the transformation W, which is represented by the matrix **R**,
(c) find the matrix **R**.

(2)

(Total for question = 16 marks)

Q3.

The transformation P is an enlargement, centre the origin, with scale factor k, where $k > 0$

The transformation Q is a rotation through angle θ degrees anticlockwise about the origin.

The transformation P followed by the transformation Q is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

(a) Determine

(i) the value of k,
(ii) the smallest value of θ

(4)

A square S has vertices at the points with coordinates (0, 0), (a, –a), (2a, 0) and (a, a) where a is a constant.

The square S is transformed to the square S' by the transformation represented by **M**.

(b) Determine, in terms of a, the area of S'

(2)

(Total for question = 6 marks)

Q4.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation *T* of the plane is represented by the matrix **M**.

The triangle *R* is transformed to the triangle *S* by the transformation *T*.

Given that the area of *S* is 63 square units,

(b) find the area of *R*.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation *T*.

(2)

(Total for question = 6 marks)

Q5.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation *M*.

Prove that, for the linear transformation *M*, there are no invariant lines.

(5)

(Total for question = 5 marks)

Q6.

*With respect to the **right-hand rule**, a rotation through $\theta°$ anticlockwise about the y-axis is represented by the matrix*

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The point *P* has coordinates (8, 3, 2)

The point *Q* is the image of *P* under the transformation reflection in the plane $y = 0$

(a) Write down the coordinates of *Q*

(1)

The point *R* is the image of *P* under the transformation rotation through 120° anticlockwise about the *y*-axis, with respect to the **right-hand rule**.

(b) Determine the exact coordinates of *R*

(2)

(c) Hence find $|\overrightarrow{PR}|$ giving your answer as a simplified surd.

(2)

(d) Show that $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ are perpendicular.

(1)

(e) Hence determine the exact area of triangle *PQR*, giving your answer as a surd in simplest form.

(2)

(Total for question = 8 marks)

Q7.

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) (i) Describe fully the single geometrical transformation *P* represented by the matrix **P**.

(ii) Describe fully the single geometrical transformation *Q* represented by the matrix **Q**.

(4)

The transformation *P* followed by the transformation *Q* is the transformation *R*, which is represented by the matrix **R**.

(b) Determine **R**.

(1)

(c) (i) Evaluate the determinant of **R**.

(ii) Explain how the value obtained in (c)(i) relates to the transformation *R*.

(2)

(Total for question = 7 marks)

Q8.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}$$

(a) Find $\mathbf{A}^{-1}$

(2)

The transformation represented by the matrix **B** followed by the transformation represented by the matrix **A** is equivalent to the transformation represented by the matrix **P**.

(b) Find **B**, giving your answer in its simplest form.

(3)

(Total for question = 5 marks)

Q9.

(i)

$$\mathbf{A} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix}$$

where p is a constant.

(a) Find, in terms of p, the matrix **AB**

(2)

Given that

$$\mathbf{AB} + 2\mathbf{A} = k\mathbf{I}$$

where k is a constant and **I** is the 2 × 2 identity matrix,

(b) find the value of p and the value of k.

(4)

(ii)

$$\mathbf{M} = \begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix}, \text{ where } a \text{ is a real constant}$$

Triangle T has an area of 15 square units.
Triangle T is transformed to the triangle T' by the transformation represented by the matrix **M**.
Given that the area of triangle T' is 270 square units, find the possible values of a.

(5)

(Total for question = 11 marks)

Q10.

$$\mathbf{M} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

(a) Show that **M** is non-singular.

(2)

The hexagon R is transformed to the hexagon S by the transformation represented by the matrix **M**.

Given that the area of hexagon R is 5 square units,

(b) find the area of hexagon S.

(1)

The matrix **M** represents an enlargement, with centre (0, 0) and scale factor k, where $k > 0$, followed by a rotation anti-clockwise through an angle θ about (0, 0).

(c) Find the value of k.

(2)

(d) Find the value of θ.

(2)

(Total for question = 7 marks)

<u>Mark Scheme</u> – Linear Tranformations

Q1.

Question	Scheme	Marks	AOs
(a)	Rotation	B1	1.1b
	120 degrees (anticlockwise) or $\frac{2\pi}{3}$ radians (anticlockwise) Or 240 degrees clockwise or $\frac{4\pi}{3}$ radians clockwise	B1	2.5
	About (from) the origin. Allow (0, 0) or O for origin.	B1	1.2
		(3)	
(b)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	B1	1.1b
		(1)	
(c)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$	A1ft	1.1b
		(2)	
(d)	$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} = \ldots$ or $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \ldots$ Note: $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2}k \\ \frac{1}{2} + \frac{\sqrt{3}}{2}k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$ can score M1 (for the matrix equation) but needs an equation to be "extracted" to score the next A1	M1	3.1a
	$-\frac{\sqrt{3}}{2} + \frac{1}{2}k = 1$ or $\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$ or $x = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$ or $y = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$ (Note that candidates may then substitute x = 1 which is acceptable)	A1ft	1.1b
	$-\frac{\sqrt{3}}{2} + \frac{1}{2}k = 1$ or $x = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \Rightarrow k = 2 + \sqrt{3}\left(\text{or } \frac{1}{2-\sqrt{3}}\right)$	A1	1.1b
	$\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$ or $y = \frac{1}{2}x + \frac{\sqrt{3}}{2}y \Rightarrow k = 2 + \sqrt{3}\left(\text{or } \frac{1}{2-\sqrt{3}}\right)$	B1	1.1b
		(4)	
			(10 marks)

Notes
(a) B1: Identifies the transformation as a rotation B1: Correct angle. Allow equivalents in degrees or radians. B1: Identifies the origin as the centre of rotation These marks can only be awarded as the elements of a **single transformation** (b) B1: Shows the correct matrix in the correct form (c) M1: Multiplies the matrices in the correct order (evidence of multiplication can be taken from 3 correct or 3 correct ft elements) A1ft: Correct matrix (follow through their matrix from part (b)) A correct matrix or a correct follow through matrix implies both marks. (d) M1: Translates the problem into a matrix multiplication to obtain at least one equation in k or in x and y A1ft: Obtains one correct equation (follow through their matrix from part (c)) A1: Correct value for k in any form B1: Checks their answer by independently solving both equations **correctly** to obtain $2+\sqrt{3}$ both times or substitutes $2+\sqrt{3}$ into the other equation to confirm its validity

Q2.

Question	Scheme		Marks	AOs
(i) (a)	Multiplies the matrix **A** by itself and sets equal to **I** to form one equation in a only and another equation involving both a and b. $\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Rightarrow 4+a(a-4)=1$ and either $2a+ab=0$ or $2(a-4)+b(a-4)=0$ or $a(a-4)+b^2=1$		M1	3.1a
	Solves a 3TQ involving only the constant a. This could come after a value of b is found and this value substituted into an equation involving both a and b $a^2-4a+3=0\Rightarrow(a-3)(a-1)=0\Rightarrow a=...$		dM1	1.1b
	$a=1, a=3$		A1	11b
	Substitutes a value for a into an equation involving both a and b and solves for b. e.g. $2(1)+(1)b\Rightarrow b=...$ $2(1-4)b+(1-4)=0\Rightarrow b=...$ $(1)(1-4)+b^2=1\Rightarrow b=...$	Alternatively uses $2a+ab=0$ $a(2+b)=0$ As $a\neq 0$ $2+b=0\Rightarrow b=...$	dM1	1.1b
	$b=-2$		A1	1.1b
			(5)	

	Alternative (i) (a) Finds $\mathbf{A}^{-1}$ in terms of a and b, sets equal to $\mathbf{A}$ and attempts to find at least two different equations. Allow a single sign slip $\frac{1}{2b-a(a-4)}\begin{pmatrix} b & -a \\ -(a-4) & 2 \end{pmatrix} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$ One equation from $\frac{b}{2b-a(a-4)}=2$, $\frac{2}{2b-a(a-4)}=b$ One equation from $\frac{-a}{2b-a(a-4)}=a$, $\frac{-(a-4)}{2b-a(a-4)}=a-4$	M1	3.1a
	Uses their value of b and their value of the determinant to form and solve a 3TQ involving only the constant a $a^2-4a+3=0$ $\Rightarrow (a-3)(a-1)=0$ $\Rightarrow a = ...$ OR Eliminates b from their equations and solve a 3TQ involving only the constant a $a^2-4a+3=0$ $\Rightarrow (a-3)(a-1)=0$ $\Rightarrow a = ...$	dM1	1.1b
	$a=1, a=3$	A1	1.1b
	$\frac{-a}{2b-a(a-4)}=a$ $\Rightarrow 2b-a(a-4)=-1 \Rightarrow \frac{b}{-1}=2$ Or $\frac{-(a-4)}{2b-a(a-4)}=a-4$ $\Rightarrow 2b-a(a-4)=-1$ $\Rightarrow \frac{2}{-1}=b$ Substitutes a value for a into an equation to find a value for b	dM1	1.1b
	$b=-2$	A1	1.1b

(b)	Uses their smallest value of a and their value for b to form two equations $\begin{pmatrix} 2 & 'a' \\ 'a-4' & 'b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2x+ay=x$ and $(a-4)x+by=y$ $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 2x+y=x$ and $-3x-2y=y$	M1	3.1a
	$2x+y=x \Rightarrow x+y=0$ o.e. and $-3x-2y=y \Rightarrow x+y=0$ o.e.	M1	1.1b
	$x+y=0$ o.e.	A1	2.1
		(3)	
(ii)(a)	Area of the triangle $T=3$	B1	1.1b
	Complete method to find a value for p. Need to see an attempt at the determinant and setting equal to 15 divided by their area of T. The resulting 3TQ needs to be solved to find a value of p. Determinant $3p\times p-(-1)\times 2p=\dfrac{15}{\text{'their area'}} \Rightarrow p=\ldots$	M1	3.1a
	$3p^2+2p-5(=0)$	A1	1.1b
	$p=1$ must reject $p=-\dfrac{5}{3}$	A1	1.1b
		(4)	
(b)	$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(c)	(their matrix found in part (b))$\begin{pmatrix} 'p' & 2'p' \\ -1 & 3'p' \end{pmatrix}=\begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$ $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}=\begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 3 & 6 \\ 2 & -6 \end{pmatrix}$	A1ft	1.1b
		(2)	
			(16 marks)

Notes:
(i)(a) **M1:** Forming two equations, one involving a only and one involving a and b **dM1:** Dependent on previous mark, solves a 3TQ involving a **A1:** Correct values for a **dM1:** Dependent on first method mark Substitutes one of their values of a into an equation involving a and b and solve to find a value for b. Alternatively factorises either $2a + ab = 0$ and uses $a \neq 0$ to find a value for b. **A1:** Correct value for b
Alternative(i)(a) **M1:** Finds $\mathbf{A}^{-1}$ and sets equal to **A** and forms two different equations **dM1:** Dependent on previous mark. Eliminates b from their equations and solves a 3TQ involving only the constant a. Alternatively if the value of b is found first substitutes their value for b into their determinant $= -1$ to form and solve a 3TQ for a **A1:** Correct value for a **dM1:** Dependent on first method mark. Substitutes a value for a into an equation to find a value for b. Alternatively uses one equation to find the determinant $= -1$ and uses this to find a value of b. **A1:** Correct values for b
(b) **M1:** Extracts simultaneous equations using their matrix **A** with their smaller value of a. **M1:** Gathers terms from their two equations. **A1:** Achieves the correct equations and deduces the correct line. Accept equivalent equations as long as both have been shown to be the same.
(ii)(a) **B1:** Area of the triangle $T = 3$ **M1:** Full method. Finds the determinant, sets equal to 15/their area and solves the resulting 3TQ **A1:** Correct quadratic **A1:** $p = 1$ only
(b) **B1** One correct row or column **B1:** All correct
(c) **M1:** Multiplies the matrices **QP** in the correct order (if answer only then evidence can be taken from 3 correct or 3 correct ft elements) **A1ft:** Correct matrix following through on their answer to part (b) and their value of p as long as it is a positive constant

Q3.

Question	Scheme	Marks	AOs
(a) Way 1	$\det \mathbf{M} = -4\times -4 - 4\sqrt{3}\times -4\sqrt{3} = ... \Rightarrow k = \sqrt{\det \mathbf{M}} = ...$	M1	3.1a
	$k = 8$	A1	1.1b
	$\Rightarrow \mathbf{Q} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = ...$	M1	1.1b
	$(\cos\theta < 0, \sin\theta > 0 \Rightarrow$ Quadrant 2 so$)$ $\theta = 120°$	A1	1.1b
		(4)	
Way 2	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$	M1	3.1a
	Achieves both the equations $k\cos\theta = -4$ and $k\sin\theta = 4\sqrt{3}$	A1	1.1b
	$\frac{k\sin\theta}{k\cos\theta} = \frac{4\sqrt{3}}{-4} \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = ...$	M1	1.1b
	$\theta = 120°$ and $k = 8$	A1	1.1b
		(4)	
(b)	Area of S' = area of $S \times k^2$ (The area of the square $S = 2a^2$)	M1	1.1b
	Area of $S' = 128a^2$	A1ft	2.2a
		(2)	
		(6 marks)	

Notes:

(a) Way 1

M1: A full method to find k such as attempting the square root of the determinant of **M**. It is immediately deducible so the method may be implied by $k = 8$.

A1: $k = 8$

M1: A full method to find a value of θ using their k, no need to justify quadrant. Only one equation needed for this mark. Allow if a radians answer is given. May be implied by a correct angle.

A1: Correct angle in degrees.

Way 2

M1: Multiplies the correct matrix representing transformation Q by the matrix representing transformation P and sets equal to matrix **M**. Allow for the matrices either way round as the transformations commute. No need to see the identity matrix, just multiplying through by k is sufficient.

A1: Both correct equations. Note that if a correct value of k is found, this A is scored under Way 1.

M1: Solves their simultaneous equations to find a value for θ (or k)

A1: $\theta = 120°$ and $k = 8$

(b)

M1: Complete method to find the area of S' : 'their k^2' $\times$ 'their $2a^2$'. Must be an attempt at the area of S but it need not be correct.

A1ft: Deduces the correct area for S', follow through their value of k

Q4.

Question	Scheme	Marks	AOs
(a)	$(\det(\mathbf{M}) =)\ (4)(-7)-(2)(-5)$	M1	1.1a
	$\mathbf{M}$ is non-singular because $\det(\mathbf{M}) = -18$ and so $\det(\mathbf{M}) \neq 0$	A1	2.4
		(2)	
(b)	Area $R = \dfrac{\text{Area } S}{(\pm)\lvert\det \mathbf{M}\rvert} = \ldots$	M1	1.2
	$\text{Area}(R) = \dfrac{63}{\lvert -18\rvert} = \dfrac{7}{2}$ oe	A1ft	1.1b
		(2)	
(c)	$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 4x-10x \\ 2x-14x \end{pmatrix}$	M1	1.1b
	$= \begin{pmatrix} -6x \\ -12x \end{pmatrix}$ and so all points on $y = 2x$ map to points on $y = 2x$, hence the line is invariant. OR $= -6\begin{pmatrix} x \\ 2x \end{pmatrix}$ hence $y = 2x$ is invariant.	A1	2.1
		(2)	
		(6 marks)	

Notes		
(a)	M1	An attempt to find $\det(\mathbf{M})$. Just the calculation is sufficient. Site of -18 implies this mark, which may be embedded in an attempt at the inverse..
	A1	$\det(\mathbf{M}) = -18$ **and** reference to zero, e.g. $-18 \neq 0$ **and** conclusion. The conclusion may precede finding the determinant (e.g. "Non-singular if $\det(\mathbf{M}) \neq 0$, $\det(\mathbf{M}) = -18 \neq 0$" is sufficient or accept "Non-singular if $\det(\mathbf{M}) \neq 0$, $\det(\mathbf{M}) = -18$, therefore non-singular" or some other indication of conclusion.) Need not mention "$\det(\mathbf{M})$" to gain both marks here, a correct calculation, statement $-18 \neq 0$, and conclusion hence $\mathbf{M}$ is non-singular can gain M1A1.
(b)	M1	Recalls determinant is needed for area scale factor by dividing 63 by ±their determinant.
	A1ft	$\dfrac{7}{2}$ or follow through $\dfrac{63}{\lvert \text{their det} \rvert}$. Must be positive and should be simplified to single fraction or exact decimal. (Allow if made positive following division by a negative determinant.)
(c)	M1	Attempts the matrix multiplication shown or with equivalent, e.g $\begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix}$. May use $\begin{pmatrix} x \\ y \end{pmatrix}$ **and** substitute $y = 2x$ later and this is fine for the method.
	A1	Correct multiplication and working leading to conclusion that the line is invariant. If the -6 is not extracted, they must make reference to image points being on line $y = 2x$. If the -6 is extracted to show it is a multiple of $\begin{pmatrix} x \\ 2x \end{pmatrix}$ followed by a conclusion "invariant" as minimum.

Alt for (c)	$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{-18}\begin{pmatrix} -7 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \frac{-1}{18}\begin{pmatrix} -7x+10x \\ -2x+8x \end{pmatrix}$	M1	1.1b
	$= \frac{-1}{18}\begin{pmatrix} 3x \\ 6x \end{pmatrix}\left(= \frac{-1}{6}\begin{pmatrix} x \\ 2x \end{pmatrix}\right) \Rightarrow b = 2a$ so points on line $y = 2x$ map to points on $y = 2x$, hence it is invariant.	A1	2.1
	Marks as per main scheme,		
Alt 2	(Since linear transformations map straight lines to straight lines…) E.g. $(1, 2)$ is on line $y = 2x$, and $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-10 \\ 2-14 \end{pmatrix}$	M1	1.1b
	$= \begin{pmatrix} -6 \\ -12 \end{pmatrix}$, which is also on the line $y=2x$, hence as (0,0) and (1,2) both map to points on $y = 2x$ (and transformation is linear) then $y = 2x$ is invariant.	A1	2.1
	Notes		
	M1 — Identifies a point on the line $y = 2x$ and finds its image under T. If (0,0) is used there must be a clear statement it is because this is on the line, but for other points accept with any line on $y = 2x$ without statement.		
	A1 — Shows the image and another point, which may be (0,0), on $y=2x$ both map to points on $y = 2x$ concludes line is invariant. Need not reference transformation being linear for either mark here.		
Alt 3	$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix} \Rightarrow \begin{matrix} 4x-5(mx+c) = X \\ 2x-7(mx+c) = mX+c \end{matrix}$ $\Rightarrow 2x-7(mx+c) = m(4x-5(mx+c))+c$ $\Rightarrow (5m^2-11m+2)x+(5m-8)c = 0$ $\Rightarrow (5m-1)(m-2) = 0 \Rightarrow m = \ldots$ Or similar work with $c = 0$ throughout.	M1	2.1
	$(5m-8 \neq 0 \Rightarrow c = 0)$ Hence $m = 2$ gives an invariant line (with $c = 0$), so $y = 2x$ is invariant.	A1	1.1b
	Notes		
	M1 — Attempts to find the equation of a general invariant line, or general invariant line through the origin (so may have $c = 0$ throughout). To gain the method mark they must progress from finding the simultaneous equations to forming a quadratic in m and solving to a value of m.		
	A1 — Correct quadratic in m found, with $m = 2$ as solution (ignore the other) and deduction that hence $y = 2x$ is an invariant line. Ignore errors in the $(5m-8)$ here as $c = 0$ is always a possible solution. No need to see $c = 0$ derived.		

Q5.

Question	Scheme		Marks	AOs
	$\begin{pmatrix}4 & -2\\5 & 3\end{pmatrix}\begin{pmatrix}x\\mx+c\end{pmatrix}=\begin{pmatrix}X\\mX+c\end{pmatrix}$ leading to an equation in x, m, c and X		M1	3.1a
	$4x-2(mx+c)=X$ and $5x+3(mx+c)=mX+c$		A1	1.1b
	$5x+3(mx+c)=m(4x-2(mx+c))+c$ leading to $5+3m=4m-2m^2$	$(3c=-2mc+c)$	M1	2.1
	$2m^2-m+5=0 \Rightarrow b^2-4ac=(-1)^2-4(2)(5)=...$	Solves $3c=-2mc+c \Rightarrow m=...$	dM1	1.1b
	Correct expression for the discriminant = $\{-39\}<0$ therefore there are no invariant lines.	$m=-1$ and shows a contradiction in $5+3m=4m-2m^2$ therefore there are no invariant lines.	A1	2.4
	Alternative $\begin{pmatrix}4 & -2\\5 & 3\end{pmatrix}\begin{pmatrix}x\\mx\end{pmatrix}=\begin{pmatrix}X\\mX\end{pmatrix}$ leading to an equation in x, m and X		M1	3.1a
	$4x-2(mx)=X$ and $5x+3(mx)=mX$		A1	1.1b
	$5x+3(mx)=m(4x-2(mx))$ leading to $5+3m=4m-2m^2$		M1	2.1
	$2m^2-m+5=0 \Rightarrow b^2-4ac=(-1)^2-4(2)(5)=...$		dM1	1.1b
	Correct expression for the discriminant = $\{-39\}<0$ therefore there are no invariant lines that pass through the origin no invariant lines.		A1	2.4
			(5)	
			(5 marks)	

Notes:

M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation.
A1: Correct equations.
M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m.
dM1: Dependent on the previous method, finds the value of the discriminant, this can be seen in an attempt to solve the quadratic using the formula.
Alternatively solves $3c=-2mc+c$ and finds a value for m
Note: If the quadratic equation in m is solved on a calculator and complex roots given this is M0 as they are not showing why there are no real roots.
A1: Correct expression for the discriminant, states <0 and draws the required conclusion.
Alternatively, correct value for m, shows a contradiction in $5+3m=4m-2m^2$ and draws the required conclusion.

Alternative
M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation.
A1: Correct equations.

M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m.
dM1: Dependent on the previous method, finds the value of the discriminant.
A1: Correct expression for the discriminant, states <0 and draws the required conclusion.

Q6.

Question	Scheme	Marks	AOs
(a)	Coordinates of Q are $(8,-3,2)$	B1	2.2a
		(1)	
(b)	Coordinates of R are $\begin{pmatrix}\cos 120° & 0 & \sin 120° \\ 0 & 1 & 0 \\ -\sin 120° & 0 & \cos 120°\end{pmatrix}\begin{pmatrix}8\\3\\2\end{pmatrix} = \ldots$ or $\begin{pmatrix}-0.5 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -0.5\end{pmatrix}\begin{pmatrix}8\\3\\2\end{pmatrix} = \ldots$	M1	1.1a
	So R is $\left(-4+\sqrt{3},\ 3,\ -4\sqrt{3}-1\right)$	A1	1.1b
		(2)	
(c)	Finds the distance $PR = \sqrt{\left(8-'\left(-4+\sqrt{3}\right)'\right)^2 + (3-'3')^2 + \left(2-'\left(-4\sqrt{3}-1\right)'\right)^2}$ Alternatively finds their $\overrightarrow{PR}$ or their $\overrightarrow{RP}$ then applies length of a vector formula. $\sqrt{\left(12-\sqrt{3}\right)^2 + \left(3+4\sqrt{3}\right)^2}$ or $\sqrt{\left(-12+\sqrt{3}\right)^2 + \left(-3-4\sqrt{3}\right)^2}$	M1	2.1
	$= \sqrt{204}\ \left(= 2\sqrt{51}\right)$ cso	A1	1.1b
		(2)	
(d)	$\overrightarrow{PR}.\overrightarrow{PQ} = \left(-12+\sqrt{3}, 0, -3-4\sqrt{3}\right).(0,-6,0) = 0$ hence perpendicular	B1ft	1.1b
		(1)	
(e)	PQ is perpendicular to PR so Area $= \frac{1}{2} \times PQ \times PR$	M1	1.1b
	$= \frac{1}{2} \times 6 \times \sqrt{204} = 6\sqrt{51}$ cso	A1	1.1b
		(2)	
		(8 marks)	

Notes:
(a)   **B1:** Coordinates of Q correctly stated, accept as a column vector.
(b)   **M1:** Correct attempt to find coordinates of R using the given matrix with $\theta = 120$. Must be multiplying in the correct way round. With no working two correct values or (-2.27, 3, -7.93) implies this mark.   **A1:** Correct exact coordinates as shown in scheme. Accept as a column vector. Cos 120 and sin 120 must have been evaluated.
(c)   **M1:** Applies the distance formula with the coordinates of P and their R. Alternatively finds the vector $\overrightarrow{PR}$ or $\overrightarrow{RP}$ then applies length of a vector formula.   **A1:** Correct answer following correct coordinates of R, must be a surd but need not be fully simplified.

(d)

B1ft: Shows the dot product is zero between the vectors $\overline{PR}$ and $\overline{PQ}$ and draws the conclusion perpendicular. Accept with $\pm$ vectors for each. Follow through as long as the vectors are of the correct form, so $\overline{PR} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$ and $\overline{PQ} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$

Note They could state if vectors $\overline{PR}$ and $\overline{PQ}$ are perpendicular then $\overrightarrow{PR}.\overrightarrow{PQ} = 0$ then shows $\overrightarrow{PR}.\overrightarrow{PQ} = 0$ this is B1

(e)

M1: Correct method for the area of the triangle, follow through on their coordinates of R and Q. May see longer methods if they do not realise the triangle is right angled.

A1: For $6\sqrt{51}$ cso following correct coordinates of R

Alternative 1

M1 Complete method to find the correct area

Finding all the lengths $|PQ| = 6$, $|PR| = \sqrt{240} = 4\sqrt{15}$, $|QR| = \sqrt{204} = 2\sqrt{51}$

Use cosine rule to find an angle e.g. $\cos PRQ = \dfrac{240+204-36}{2\times\sqrt{240}\times\sqrt{204}} = \dfrac{\sqrt{85}}{10}$

leading to $PRQ = 22.7...$ or $\sin PRQ = \sqrt{1-\left(\dfrac{\sqrt{85}}{10}\right)^2} = ...\left\{\dfrac{\sqrt{15}}{10}\right\}$

Uses the area of the triangle $= \dfrac{1}{2}\times\sqrt{240}\times\sqrt{204}\times\dfrac{\sqrt{15}}{10}$ or $= \dfrac{1}{2}\times\sqrt{240}\times\sqrt{204}\times\sin 22.8$

A1: For $6\sqrt{51}$

Alternative 2

M1: Uses $\dfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$ to find the required area

e.g. $QP = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$ $RP = \begin{pmatrix} 12-\sqrt{3} \\ 0 \\ 3+4\sqrt{3} \end{pmatrix}$ cross product

$$\begin{vmatrix} 0 & 6 & 0 \\ 12-\sqrt{3} & 0 & 3+4\sqrt{3} \end{vmatrix} = -6\left(12-\sqrt{3}\right)\mathbf{i} + 6\left(3+4\sqrt{3}\right)\mathbf{k}$$

$$\text{Area } = \frac{1}{2}\sqrt{\left(-6\left(12-\sqrt{3}\right)^2 + \left(6\left(3+4\sqrt{3}\right)^2\right)\right)} = \frac{1}{2}\sqrt{7344}$$

A1: For $6\sqrt{51}$

Q7.

Question	Scheme	Marks	AOs
(a)(i)	Rotation	B1	1.1b
	90 degrees anticlockwise about the origin	B1	1.1b
(ii)	Stretch	B1	1.1b
	Scale factor 3 parallel to the y-axis	B1	1.1b
		(4)	
(b)	$\mathbf{QP} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$	B1	1.1b
		(1)	
(c)(i)	$\lvert\mathbf{R}\rvert = 3$	B1ft	1.1b
(ii)	The **area** scale factor of the transformation	B1	2.4
		(2)	
			(7 marks)

Notes

(a)(i)
B1: Identifies the transformation as a rotation
B1: Correct angle (allow equivalents in degrees or radians), direction and centre the origin
(ii)
B1: Identifies the transformation as a stretch
B1: Correct scale factor and parallel to/in/along the y-axis/y direction
(b)
B1: Correct matrix
(c)(i)
B1ft: Correct value for the determinant (follow through their **R**)
(ii)
B1: Correct explanation, must include **area**
Note: scale factor of the transformation is B0

Q8.

Question Number	Scheme		Marks
	$\mathbf{A}=\begin{pmatrix}2&-1\\4&3\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}3&6\\11&-8\end{pmatrix}$		
(a)	$\mathbf{A}^{-1}=\frac{1}{10}\begin{pmatrix}3&1\\-4&2\end{pmatrix}$	Either $\frac{1}{10}$ or $\begin{pmatrix}3&1\\-4&2\end{pmatrix}$	M1
		Correct matrix seen.	A1
			[2]
(b) **Way 1**	$\mathbf{P}=\mathbf{AB}$ $\Rightarrow\ \mathbf{A}^{-1}\mathbf{P}=\mathbf{A}^{-1}\mathbf{AB}\ \Rightarrow\ \mathbf{B}=\mathbf{A}^{-1}\mathbf{P}$		
	$\mathbf{B}\quad=\frac{1}{10}\begin{pmatrix}3&1\\-4&2\end{pmatrix}\begin{pmatrix}3&6\\11&-8\end{pmatrix}$	Multiplies **their** $\mathbf{A}^{-1}$ by $\mathbf{P}$ in correct order. This substituted statement is sufficient.	M1
	$=\begin{pmatrix}2&1\\1&-4\end{pmatrix}$	At least 2 elements correct or $k\begin{pmatrix}20&10\\10&-40\end{pmatrix}$ oe. May be unsimplified	A1
		Correct simplified matrix.	A1
			[3]
(b) **Way 2**	$\{\mathbf{P}=\mathbf{AB}\Rightarrow\}$ $\begin{pmatrix}3&6\\11&-8\end{pmatrix}=\begin{pmatrix}2&-1\\4&3\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$		
	$\begin{pmatrix}3&6\\11&-8\end{pmatrix}=\begin{pmatrix}2a-c&2b-d\\4a+3c&4b+3d\end{pmatrix}$	Attempt to multiply **A** by **B** in the correct order and puts equal to **P**	M1
	$\Rightarrow a=2,\ c=1,\ b=1,\ d=-4$		
	So, $\mathbf{B}=\begin{pmatrix}2&1\\1&-4\end{pmatrix}$	At least 2 elements are correct.	A1
		Correct matrix.	A1
			[3] **5**

Q9.

Question Number	Scheme		Marks
(i)	$\mathbf{A}=\begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix}$, $\mathbf{M}=\begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix}$	p, a are constants.	
(a)	$\{\mathbf{AB}\}=\begin{pmatrix} -5p+12 & 4p-10 \\ -15+6p & 12-5p \end{pmatrix}$	At least 2 elements are correct. Correct matrix.	M1 A1
			[2]
(b)	$\{\mathbf{AB}+2\mathbf{A}=k\mathbf{I}\}$ $\begin{pmatrix} -5p+12 & 4p-10 \\ -15+6p & 12-5p \end{pmatrix}+2\begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}=k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -3p+12 & 4p-6 \\ -9+6p & 12-3p \end{pmatrix}=\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	If 'simultaneous equations' used, allocate marks as below.	
	$"4p-10"+4=0$ or $"-15+6p"+6=0$ or $"-9+6p"="4p-6"$	Forms an equation in p	M1
	$\Rightarrow p=\frac{3}{2}$	$p=\frac{3}{2}$ o.e.	A1
	$k=-5\left(\frac{3}{2}\right)+12+2\left(\frac{3}{2}\right)\Rightarrow k=\ldots$	Substitutes their $p=\frac{3}{2}$ into "their $(-5p+12)$"$+2p$ to find a value for k or eliminates p to find k.	M1
	$k=\frac{15}{2}$	$k=\frac{15}{2}$ oe	A1
			[4]
(ii) **Way 1**	$\pm\frac{270}{15}$ $\{=\pm 18\}$	Can be implied from calculations.	B1
	$\det\mathbf{M}=(a)(2)-(-9)(1)$	Applies $ad-bc$ to $\mathbf{M}$. Require clear evidence of correct formula being used for M1 if errors seen.	M1
	$\Rightarrow 2a+9=18$ or $2a+9=-18$	Equates their $\det\mathbf{A}$ to either 18 or -18	M1
	$\Rightarrow a=4.5$ or $a=-13.5$	At least one of either $a=4.5$ or $a=-13.5$	A1
		Both $a=4.5$ and $a=-13.5$	A1
			[5]
(ii) **Way 2**	Consider vertices of triangle with area 15 units e.g. (0,0),(15,0) and (0,2) and attempting 2 values of a.		B1
	e.g. $\begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 15 & 0 \\ 0 & 0 & 2 \end{pmatrix}=\begin{pmatrix} 0 & 15a & -18 \\ 0 & 15 & 4 \end{pmatrix}$	Pre-multiplies their matrix by $\mathbf{M}$ and obtains single matrix	M1
	e.g. $\frac{1}{2}\begin{vmatrix} 0 & 15a & -18 & 0 \\ 0 & 15 & 4 & 0 \end{vmatrix}=270$	Equates their determinant to 270 and attempts to solve.	M1
	$\Rightarrow a=4.5$ or $a=-13.5$	At least one of either $a=4.5$ or $a=-13.5$	A1
		Both $a=4.5$ and $a=-13.5$	A1
			[5] **11**

Q10.

Question	Scheme	Marks	AOs
(a)	$\det(\mathbf{M}) = (1)(1) - (\sqrt{3})(-\sqrt{3})$	M1	1.1a
	$\mathbf{M}$ is non-singular because $\det(\mathbf{M}) = 4$ and so $\det(\mathbf{M}) \neq 0$	A1	2.4
		(2)	
(b)	$\text{Area}(S) = 4(5) = 20$	B1ft	1.2
		(1)	
(c)	$k = \sqrt{(1)(1) - (\sqrt{3})(-\sqrt{3})}$	M1	1.1b
	$= 2$	A1ft	1.1b
		(2)	
(d)	$\cos\theta = \frac{1}{2}$ or $\sin\theta = \frac{\sqrt{3}}{2}$ or $\tan\theta = \sqrt{3}$	M1	1.1b
	$\theta = 60°$ or $\frac{\pi}{3}$	A1	1.1b
		(2)	
	(7 marks)		

		Question Notes
(a)	M1	An attempt to find $\det(\mathbf{M})$.
	A1	$\det(\mathbf{M}) = 4$ **and** reference to zero, e.g. $4 \neq 0$ **and** conclusion.
(b)	B1ft	20 or a correct ft based on their answer to part (a).
(c)	M1	$\sqrt{(\text{their }\det\mathbf{M})}$
	A1ft	2
(d)	M1	**Either** $\cos\theta = \frac{1}{(\text{their } k)}$ **or** $\sin\theta = \frac{\sqrt{3}}{(\text{their } k)}$ **or** $\tan\theta = \sqrt{3}$
	A1	$\theta = 60^{\circ}$ or $\frac{\pi}{3}$. Also accept any value satisfying $360n + 60^{\circ}$, $n \in \mathbb{Z}$, o.e.