Confidence Intervals and Tests Using *t* - Distribution

Questions

Q1.

A doctor believes that a four-week exercise programme can reduce the resting heart rate of her patients. She takes a random sample of 7 patients and records their resting heart rate before the exercise programme and again after the exercise programme.

Patient	*A*	*B*	*C*	*D*	*E*	*F*	*G*
Resting heart rate before	65	68	77	79	80	88	92
Resting heart rate after	63	65	73	76	80	84	80

(a) Using a 5% level of significance, carry out an appropriate test of the doctor's belief. You should state your hypotheses, test statistic and critical value.

(7)

(b) State the assumption made about the resting heart rates that was required to carry out the test.

(1)

(Total for question = 8 marks)

Q2.

A machine is set to fill pots with yoghurt such that the mean weight of yoghurt in a pot is 505 grams.

To check that the machine is working properly, a random sample of 8 pots is selected. The weight of yoghurt, in grams, in each pot is as follows

508 510 500 500 498 503 508 505

Given that the weights of the yoghurt delivered by the machine follow a normal distribution with standard deviation 5.4 grams,

(a) find a 95% confidence interval for the mean weight, μ grams, of yoghurt in a pot.

Give your answers to 2 decimal places.

(4)

(b) Comment on whether or not the machine is working properly, giving a reason for your answer.

(1)

(c) State the probability that a 95% confidence interval for μ will not contain μ grams.

(1)

(d) Without carrying out any further calculations, explain the changes, if any, that would need to be made in calculating the confidence interval in part (a) if the standard deviation was unknown. Give a reason for your answer.
You may assume that the weights of the yoghurt delivered by the machine still follow a normal distribution.

(2)

(Total for question = 8 marks)

Q3.

Jemima makes jam to sell in a local shop. The jam is sold in jars and the weight of jam in a jar is normally distributed.

Jemima takes a random sample of 8 of her jars of jam and weighs the contents of each jar, x grams. Her results are summarised as follows

$$\sum x = 3552 \qquad \sum x^2 = 1\,577\,314$$

(a) Calculate a 95% confidence interval for the mean weight of jam in a jar.

(5)

The labels on the jars state that the average contents weigh 440 grams.

(b) State, giving a reason, whether or not Jemima should be concerned about the labels on her jars of jam.

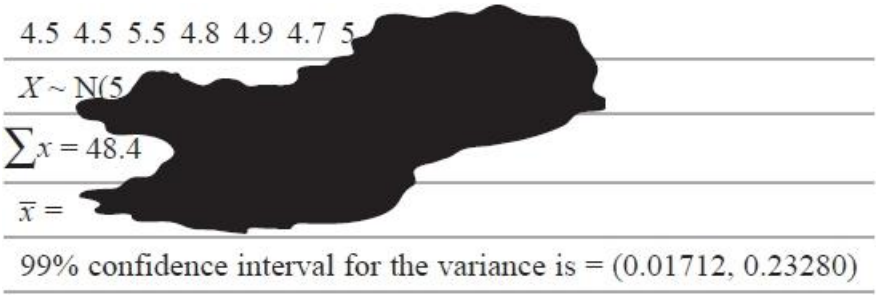
(1)

(Total for question = 6 marks)

Q4.

A company manufactures bolts. The diameter of the bolts follows a normal distribution with a mean diameter of 5 mm.

Stan believes that the mean diameter of the bolts is less than 5 mm. He takes a random sample of 10 bolts and measures their diameters. He calculates some statistics but spills ink on his work before completing them. The only information he has left is as follows



Stating your hypotheses clearly, test, at the 5% level of significance, whether or not Stan's belief is supported.

(9)

(Total for question = 9 marks)

Q5.

Jamland and *Goodjam* are two suppliers of jars of jam. The weights of the jars of jam produced by each supplier can be assumed to be normally distributed with unknown, but equal, variances. A random sample of 20 jars of jam is taken from those supplied by *Jamland*.

Based on this sample, the 95% confidence interval for the mean weight of a jar of *Jamland* jam, in grams, is

[492, 507]

A random sample of 10 jars of jam is selected from those supplied by *Goodjam*. The weight of each jar of *Goodjam* jam, *y* grams, is recorded. The results are summarised as follows

$$\bar{y} = 480 \qquad s_y^2 = 280$$

Find a 90% confidence interval for the value by which the mean weight of a jar of jam supplied by *Jamland* exceeds the mean weight of a jar of jam supplied by *Goodjam*.

(11)

(Total for question = 11 marks)

Q6.

A new employee, Kim, joins an existing employee, Jiang, to work in the quality control department of a company producing steel rods.
Each day a random sample of rods is taken, their lengths measured and a 95% confidence interval for the mean length of the rods, in metres, is calculated. It is assumed that the lengths of the rods produced are normally distributed.

Kim took a random sample of 25 rods and used the *t* distribution to obtain a 95% confidence interval of (1.193, 1.367) for the mean length of the rods.
Jiang commented that this interval was a little wider than usual and explained that they usually assume that the standard deviation does not change and can be taken as 0.175 metres.

(a) Test, at the 10% level of significance, whether or not Kim's sample suggests that the standard deviation is different from 0.175 metres. State your hypotheses clearly.

(9)

Using Kim's sample and the normal distribution with a standard deviation of 0.175 metres,

(b) find a 95% confidence interval for the mean length of the rods.

(3)

(Total for question = 12 marks)

Q7.

The times, *x* seconds, taken by the competitors in the 100m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

	No. of competitors	Sample mean $\bar{x}$	$\sum x^2$
Girls	8	83.1	55 746
Boys	7	88.9	56 130

Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly.

(7)

(b) Stating your hypotheses clearly, test the mother's claim. Use a 5% level of significance.

(6)

(Total for question = 13 marks)

Q8.

Alexa believes that students are equally likely to achieve the same percentage score on each of
two tests, paper I and paper II. She randomly selects 8 students and gives them each paper I
and paper II. The percentage scores for each paper are recorded.

The following paired data are collected.

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Paper I (%)	70	70	84	80	64	65	65	90
Paper II (%)	64	76	72	74	68	64	58	76

Test, at the 1% significance level, whether or not there is evidence to support Alexa's belief. State your hypotheses clearly and show your working.

(7)

(Total for question = 7 marks)

Q9.

The times taken by children to run 150m are normally distributed. The times taken, x seconds, by a random sample of 9 boys and an independent random sample of 6 girls are recorded. The following statistics are obtained.

	Number of children	Sample mean $\bar{x}$	$\sum x^2$
Boys	9	22.8	4693.60
Girls	6	29.5	5236.12

(a) Test, at the 10% level of significance, whether or not the variances of the two distributions are equal. State your hypotheses clearly.

(7)

The Headteacher claims that the mean time taken for the girls is more than 5 seconds greater than the mean time taken for the boys.

(b) Stating your hypotheses clearly, test the Headteacher's claim. Use a 1% level of significance and show your working clearly.

(7)

(Total for question = 14 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	d: 2 3 4 3 0 4 12	M1	3.1b
	$\bar{d} = \pm 4 \quad s_d = = \sqrt{\frac{1}{6}('198' - 7('4')^2)} = \sqrt{14.333\ldots} = 3.7859\ldots$	M1	1.1b
	$H_0 : \mu_d = 0 \quad H_1 : \mu_d > 0$	B1	2.5
	$t = \pm \frac{"\pm 4"}{\frac{"3.78..."}{\sqrt{7}}}$	M1	1.1b
	$= \pm$ 2.795… …. awrt ± 2.80	A1	1.1b
	Critical value $t_6 = \pm 1.943$	B1	1.1b
	[2.80 > 1.943,] therefore there is sufficient evidence to support the doctor's belief/evidence to suggest resting heart rate is reduced.	A1	2.2b
		(7)	
(b)	**Differences** in resting heart rates must be normally distributed for the test to be valid.	B1	2.4
		(1)	
			(8 marks)

Notes	
(a)	**M1:** For understanding paired t-test is required and attempting to find differences at least 5 correct, allow $\pm$ (may be implied by correct $\bar{d}$ and s_d) **M1:** Complete method for $\bar{d}$ and s_d or $(s_d)^2$ **B1:** Correct model for differences with both hypotheses correct in terms of μ / μ_d (sign of H_1 must be compatible with their $\bar{d}$) **M1:** Method for finding test statistic with their values **A1:** awrt ± 2.80 (allow awrt ± 2.8 from correct working) **B1:** Correct critical ± 1.943 (or better) with compatible sign **A1:** Correct comparison to deduce that the doctor's belief is supported. Must be consistent with their CV and their test statistic (dependent upon all M marks). SC: Difference of means test apply scheme but also allow 2nd B1 for $t_{12} = 1.782$
(b)	**B1:** Correct modelling assumption

Q2.

Question	Scheme	Marks	AOs
(a)	Mean = 504	B1	1.1b
	1.96	B1	3.3
	$504 \pm \frac{5.4}{\sqrt{8}} \times$"1.96"	M1	2.1
	(500.258, 507.742)	A1	1.1b
		(4)	
(b)	**505** is in the confidence **interval** therefore there is evidence that the machine is **working** properly	B1ft	2.2b
		(1)	
(c)	5% oe	B1	1.1b
		(1)	
(d)	s needs to be used instead of σ **and** a t-value instead of the z value	B1	3.3
	since the sample is small therefore you can't use the normal distribution	B1	3.5b
		(2)	
			(8 marks)

Notes:		
(a)	B1	504 may be seen in part(b)
	B1	For realising a normal distribution must be used as a model and finding the correct value 1.96
	M1	For $504 \pm \frac{5.4}{\sqrt{8}} \times$"$z$ value". $\lvert z \rvert > 1$ May be implied by a correct CI
	A1	awrt 500.26 and 507.74 NB using t gives 500.29 and 507.71
(b)	B1ft	Drawing a correct inference (ft) using their answer to part (a) and the 505 from the question. Reason must be given. Ignore incorrect non – contextual
(c)	B1	5%
(d)	B1	create new model by using s and t. Allow if state use CI $\mu \pm \frac{s}{\sqrt{n}} \times$"$t$" or use $s = 4.44$ and $t = 2.365$
	B1	For recognising that the sample is small

Q3.

Qu	Scheme	Marks	Grade	AO
(a)	$\overline{x} = 444$	M1	Low	2.1
	$s_x^2 = \frac{1577\,314 - 8\times 444^2}{7} = \frac{226}{7} = 32.2857..$	A1	Low	1.1b
	$t_7(5\%)$ 2-tail cv = 2.365	B1	Low	1.1b
	95% CI for μ is: $444 \pm 2.365\times\sqrt{\frac{32.2857...}{8}}$	M1	Med	2.1
	$= (439.248..., 448.75...)$ = awrt **(439, 449)**	A1 (5)	Med	1.1b
(b)	440 is in CI so the average contents statement is OK	B1 (1)	High	2.2b
		(6 marks)		

	Notes
(a)	1st M1 for finding mean and attempting s^2 1st A1 for correct mean and s^2 (accept awrt 3sf) B1 for a correct cv of 2.365 or better 2nd M1 for use of correct formula, ft their mean, s_x and cv for t (use of 1.96 is M0) 2nd A1 for awrt (439,449)
(b)	1st B1 for correct statement about 440 and interval and conclusion

Q4.

Question	Scheme	Marks	AOs
	99% confidence interval for Var uses χ^2 values of 1.735 or 23.589	B1	3.3
	$\frac{9s^2}{1.735}=0.2328$ or $\frac{9s^2}{23.589}=0.01712$	M1	2.1
	$s^2=\frac{0.2328\times"1.735"}{9}$ or $\frac{0.01712\times"23.589"}{9}$ $[=0.04487...]$	dM1	1.1b
	$\bar{x}=4.84$	B1	1.1b
	$H_0:\mu=5$ $H_1:\mu<5$	B1	2.5
	CV $t_9=-1.833$	B1	1.1b
	$t=\pm\frac{"4.84"-5}{\sqrt{"0.0449"/10}}$	M1	1.1b
	$=$ awrt -2.39	A1	1.1b
	Stan's belief is supported **or** there is evidence that the **mean diameter** of the bolts is less than **5mm**	A1ft	2.2b
		(9)	
			(9 marks)

Notes:
B1: For realising a χ^2 distribution must be used as a model and finding a correct value
M1: For realising the need to set $\frac{9s^2}{"\text{smallest }\chi^2"}=0.2328$ or $\frac{9s^2}{"\text{largest }\chi^2"}=0.01712$
dM1: correct method used to solve equation to find s^2
B1: awrt 4.84
B1: Both hypotheses correct using the notation μ
B1: $\pm$ 1.833
M1: For us of correct formula ie $\pm\frac{"\text{their }4.84"-5}{\sqrt{"\text{their }0.0449"/10}}$ If "4.84" not shown it must be correct here
A1: -2.39
A1ft: Drawing a correct inference following through their CV and test statistic (must have matching signs)
NB if chi squared values not shown $s^2=0.045$ or 0.0449 award B0 M1M1 for awrt 0.04487 award B1 M1 A1
Use of $2(2.5758)\frac{\sigma}{\sqrt{10}}=0.21568$ gives $\sigma=\sqrt{0.0175}$ could get B0M0M0B1B1B1M0A0A0 Unless continue to get $s^2=\frac{10}{9}0.0175=0.0194...$
Use of $2(1.833)\frac{s}{\sqrt{10}}=0.21568$ gives $s=0.1860$ could get B0M0M0B1B1B1M1A0A1

Q5.

Question Number	Scheme	Marks
	$\bar{x} = \frac{492+507}{2}$	M1
	$= 499.5$	A1cao
	$2.093\frac{s}{\sqrt{20}} = 7.5$	M1,B1
	$s = 16.02533...$ $(s^2 = 256.81..6)$	A1
	$s_p^2 = \frac{19\times 16.025..^2 + 9\times 280}{28} = 264.26...$	M1A1ft
	$t_{28(0.05)} = 1.701$	B1
	90% CI $= (499.5 - 480) \pm 1.701 \times \sqrt{264.26} \times \sqrt{\frac{1}{20}+\frac{1}{10}}$	M1A1ft
	$= (8.8, 30.2)$	A1cao (11)

	Notes	
	M1 $\bar{x} = \frac{492+507}{2}$ A1 499.5 cao M1 $t-$value$\frac{s}{\sqrt{20}} = 7.5$ B1 2.093 A1 awrt 16.0 for s or 257 for s^2 M1 $\frac{"n_1-1"\times(s \text{ or } s^2)+"n_2-1"\times(s \text{ or } s^2)}{n_1+n_2-2}$ finding s_p^2 A1 ft their s^2 B1 awrt 1.701 M1 $(\bar{x}-480) \pm t$-value $\times \sqrt{s_p^2} \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$ A1 ft their s_p^2 and $\bar{x}$ A1cao awrt 8.8 and awrt 30.2	

Q6.

Qu	Scheme	Marks		AO
(a)	From CI $\bar{x} = \frac{1.193+1.367}{2} = 1.28$	B1		1.1b
	or width = 1.367 − 1.193 = 0.174			
	$1.367 = "1.28" \pm 2.064 \times \frac{s}{\sqrt{25}}$ or $"0.174" = 2 \times 2.064 \times \frac{s}{\sqrt{25}}$	M1;A1		3.4 1.1b
	$\Rightarrow s = 0.210755...$	A1		1.1b
	$H_0 : \sigma = 0.175$ $H_1 : \sigma \neq 0.175$	B1		2.5
	$\chi_{24}^2 = \frac{24s^2}{\sigma^2} =$, 34.8092... awrt 34.8	M1, A1		3.3 1.1b
	$\chi_{24}^2(10\%)$ 2-tail CR is $\chi_{24}^2 < \underline{13.848}$ or $\chi_{24}^2 > \underline{36.415}$	B1		2.1
	34.8 is not significant so insufficient evidence that $\sigma \neq 0.175$	A1		2.2b
		(9)		
(b)	$"1.28" \pm z \times \frac{0.175}{\sqrt{25}}$	M1		3.3
	$z = 1.96$	B1		1.1b
	= (1.211..., 1.349...) = awrt **(1.21, 1.35)**	A1		1.1b
		(3)		
		(12 marks)		

	Notes
(a)	1st B1 for finding mean from CI or calculation of width of CI 1st M1 for using the given t model to form an equation in s. (Allow t for 2.064 where $2 < t < 3$) 1st A1 for correct use of $t_{24} = 2.064$ 2nd A1 for $s = 0.21$ or better 2nd B1 for correct hypotheses in terms of σ. 2nd M1 for selecting the appropriate model for this test 3rd A1 for test statistic awrt 34.8 3rd B1 for at least one correct critical value 4th A1 for a correct conclusion confirming that assuming st. dev = 0.175 is OK
(b)	M1 for use of correct formula with $1.6 < z < 2$ (ft $\bar{x}$ if found in (a)) B1 for $z = 1.96$ or better used A1 for an interval awrt (1.21, 1.35)

Q7.

Question	Scheme	Marks	AOs
(a)	H_0: $\sigma_G^2 = \sigma_B^2$, H_1: $\sigma_G^2 \neq \sigma_B^2$,	B1	2.5
	$s_B^2 = \frac{1}{6}(56130 - 7 \times 88.9^2) = \frac{807.53}{6} = 134.6$	M1 A1	2.1 1.1b
	$s_G^2 = \frac{1}{7}(55746 - 8 \times 83.1^2) = \frac{501.12}{7} = 71.58$	A1	1.1b
	$\frac{s_B^2}{s_G^2} = 1.880\ldots$	M1	3.4
	critical value $F_{6,7} = 3.87$	B1	1.1b
	not significant, variances can be treated as the same	A1 ft	2.2b
		(7)	
(b)	H_0: $\mu_B = \mu_G$, H_1: $\mu_B > \mu_G$	B1	2.5
	pooled estimate of variance $s^2 = \frac{6 \times 134.6 + 7 \times 71.58}{13} = 100.6653\ldots$	M1	3.1b
	test statistic $t = \frac{88.9 - 83.1}{s\sqrt{\frac{1}{7} + \frac{1}{8}}}$ = awrt 1.12	M1 A1	1.1b 1.1b
	critical value $t_{13}(5\%) = 1.771$	B1	1.1b
	Insufficient evidence to support mother's claim	A1 ft	2.2b
		(6)	
			(13 marks)

	Notes
(a)	**B1:** Both hypotheses correct using the notation σ^2. Allow σ rather than σ^2. **M1:** Using a correct Method for either s_B^2 or s_G^2 May be implied by a correct value **A1:** awrt 135 **A1:** awrt 71.6 **M1:** Using the F-distribution as the model eg $\frac{s_B^2}{s_G^2}$ **B1:** awrt 3.87 **A1ft:** Drawing a correct inference following through their CV and value for $\frac{s_B^2}{s_G^2}$
(b)	**B1:** Both hypotheses correct using the notation μ. **M1:** For realising the need to find the pooled estimate for the test require from a correct interpretation of the question. **M1:** Correct method for test statistic $t = \frac{88.9 - 83.1}{\text{"their } s\text{"}\sqrt{\frac{1}{7} + \frac{1}{8}}}$ May be implied by a correct awrt 1.12 **A1:** awrt 1.12 **B1:** awrt 1.77 **A1ft** Drawing a correct inference following through their CV and test statistic

Q8.

Question	Scheme	Marks	AOs
	d: 6 –6 12 6 –4 1 7 14	M1	3.1b
	$\bar{d} = \pm 4.5$ $\quad s_d = \sqrt{50.285...} = 7.09\ldots$	M1	1.1b
	$H_0 : \mu_d = 0$ $\quad H_1 : \mu_d \neq 0$	B1	3.3
	$t = \pm \frac{"4.5"\sqrt{8}}{"7.09..."}$ oe	M1	1.1b
	$= \pm\ 1.7948\ldots.$ awrt $\pm 1.79/1.8$	A1	1.1b
	Critical value $t_7 = \pm 3.499$	B1	1.1b
	There is insufficient evidence that the **papers** are of a different level of difficulty or **Alexa's** belief is correct	A1ft	2.2b
		(7)	
			(7 marks)

Notes:
M1: for realising that the model to use is the paired t-test and finding the differences $(\pm)$ At least 3 correct
M1: correct method for finding $\bar{d}$ and s_d.
B1: Using a correct model for difference and both hypotheses correct using the notation μ_d or μ Condone $\mu_I = \mu_{II}$ and $\mu_I \neq \mu_{II}$
M1: Using the correct method to find test statistics ie $t = \pm \frac{"\text{their } 4.5"\sqrt{8}}{"\text{their } 7.09..."}$
A1: awrt1.79 or 1.8
B1: for correct critical value $t = \pm 3.499$ with compatible sign
A1ft: Drawing a correct inference in context using their CV and their value of t
NB difference of means test gets M0M0B1M0A0B0A0

Q9.

Question Number	Scheme	Marks
(a)	H_0: $\sigma_B^2 = \sigma_G^2$, H_1: $\sigma_B^2 \neq \sigma_G^2$,	B1
	$[s_B^2 =]\frac{1}{8}(4693.6 - 9 \times 22.8^2) = 1.88$	M1
	$[s_G^2 =]\frac{1}{5}(5236.12 - 6 \times 29.5^2) = 2.924$ awrt 2.92	A1
	$\frac{s_G^2}{s_B^2} = 1.555\ldots[0.643]$	M1 A1
	critical value $F_{5,8} = 3.69[0.271]$	B1
	not significant, variances are the same	A1 cso (7)
(b)	H_0: $\mu_G = \mu_B + 5$, H_1: $\mu_G > \mu_B + 5$	B1
	pooled estimate of variance $s_p^2 = \frac{8\times1.88+5\times2.924}{13} = 2.2815\ldots$ or $s_p = 1.51046\ldots$	M1
	test statistic $t = \pm\left(\frac{29.5-22.8-5}{s\sqrt{\frac{1}{9}+\frac{1}{6}}}\right) = \pm$ awrt 2.14 or $p = 0.0262$	M1 M1A1
	critical value $t_{13}(1\%) = \pm 2.650$ or $0.0262 > 0.01$	B1
	Insufficient evidence to support **Headteacher**'s claim or The **time** taken for girls is **not more than 5** seconds greater than for boys	A1 cso (7)

	Notes	Total 14
(a)	B1 both hypotheses. Must use σ or σ^2 and make clear which is H_0 and which is H_1. Do not allow in words M1 correct method for either s_B^2 or s_G^2 A1 Both s_B^2 and s_G^2 correct to 3sf allow sd's M1 allow use of s_B and s_G instead of s_B^2 or s_G^2 A1 awrt 1.56 or 0.643 B1 correct CV for their F or a correct comparison if use p A1 cso – All previous marks must be awarded. Variances are the same or var are not different	
(b)	B1 both hypotheses using μ. Do not allow $\geq$ sign instead of >. May use different letters eg A and B but they must be defined. M1 only allow use of s_B and s_G instead of s_B^2 or s_G^2 - May be seen in part(a) M1 use of correct formula with their s_p – condone missing 5 M1 use of correct formula with their s_p. (which must have been attempted) B1 correct CV but must match t-value or a correct comparison if use p A1 A correct statement with either the word Headteacher/Teacher/Head or time and not more than 5 oe do not allow contradicting statements.	