Matrices

Questions

Q1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

(2)

(b) Solve the simultaneous equations

$$2x + y - 3z = -4$$

$$4x - 2y + z = 9$$

$$3x + 5y - 2z = 5$$

(2)

(c) Interpret the answer to part (b) geometrically.

(1)

(Total for question = 5 marks)

Q2.

(In this question you must show all stages of your working.)

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

(Total for question = 8 marks)

Q3.

The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

(Total for question = 12 marks)

Q4.

M = $\begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix}$ where a is a constant

(a) Show that **M** is non-singular for all values of a.

(2)

(b) Determine, in terms of a, $\mathbf{M}^{-1}$

(4)

(Total for question = 6 marks)

Q5.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$$

where k is a constant.

(a) Determine, in simplest form in terms of k, the matrix **M N**.

(2)

(b) Given that $k = 5$

(i) write down **M N**
(ii) hence write down $\mathbf{M}^{-1}$

(2)

(c) Solve the simultaneous equations

$$\begin{aligned} 2x + y + 4z &= 2 \\ 5x + 2y - 2z &= 3 \\ 4x + y - 2z &= -1 \end{aligned}$$

(2)

(d) Interpret the answer to part (c) geometrically.

(1)

(Total for question = 7 marks)

Q6.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{pmatrix}$$

Given that **I** is the 3 × 3 identity matrix,

(a) (i) show that there is an integer k for which

$$\mathbf{AB} - 3\mathbf{C} + k\mathbf{I} = \mathbf{0}$$

stating the value of k

(ii) explain why there can be no constant m such that

$$\mathbf{BA} - 3\mathbf{C} + m\mathbf{I} = \mathbf{0}$$

(4)

(b) (i) Show how the matrix **C** can be used to solve the simultaneous equations

$$-5x + 2y + z = -14$$
$$4x\ x + 3y + 8z = 3$$
$$-6x + 11y + 2z = 7$$

(ii) Hence use your calculator to solve these equations.

(3)

(Total for question = 7 marks)

Q7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix **M** has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$

$$3x - 6y + 4z = 1$$

$$3x + 2y - z = 0$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$

$$3x - 5y + 4z = q$$

$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

(Total for question = 11 marks)

Q8.

(i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

(2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda\mathbf{I}$$

where a, b and λ are constants,

(a) determine

- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

(3)

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of θ for which the matrix **M** is singular.

(4)

(Total for question = 9 marks)

Mark Scheme - Matrices

Q1.

Question	Scheme	Marks	AOs
(a)	$\mathbf{M}^{-1} = \frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(b)	$\frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} = \ldots$	M1	1.1b
	$x = 2, y = 1, z = 3$ or $(2,1,3)$ or $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ or $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$	A1	1.1b
		(2)	
(c)	The **point** where three **planes** meet	B1ft	2.2a
		(1)	
			(5 marks)

Notes
(a) B1: Evidence that the determinant is ± 69 (may be implied by their matrix e.g. where entries are not in exact form: $\pm\begin{pmatrix} 0.014 & 0.188 & 0.072 \\ -0.159 & -0.072 & 0.203 \\ -0.377 & 0.101 & 0.116 \end{pmatrix}$)(Should be mostly correct) **Must be seen in part (a).** B1: Fully correct inverse with all elements in **exact** form (b) M1: Any complete method to find the values of x, y and z (Must be using **their inverse** if using the method in the main scheme) A1: Correct coordinates A solution not using the inverse requires a complete method to find values for x, y and z for the method mark. Correct coordinates only scores both marks. (c) B1: Describes the correct geometrical configuration. Must include the two ideas of **planes** and **meet in a point** with no contradictory statements. This is dependent on having obtained a unique point in part (b)

Q2.

Question	Scheme	Marks	AOs
(a)(i)	x / C = number of **Construction** students y / D = number of **Design** students z / H = number of **Hospitality** students	B1	3.3
(ii)	The increase in number of students in 2020 $1110 \times 0.0027\{= 2.997 \approx 3\}$ Or The number of students in 2020 $1110 \times 1.0027 = \{1112.997 \approx 1113\}$	M1	1.1b
	$x + y + z = 1110$ $C + D + H = 1110$ $x - z = 370$ o.e. $C - H = 370$ o.e. $0.0125C + 0.025D - 0.02H = 3$or 2.997 o.e $1.0125C + 1.025D + 0.98H = 1113$ or 1112.997o.e. $0.0125x + 0.025y - 0.02z = 3$ or 2.997o.e$1.0125x + 1.025y + 0.98z = 1113$ or 1112.997 o.e.	M1 A1	3.3 1.1b
		(4)	
(b)	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix}$	M1 A1ft	1.1b 1.1b
	$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$ or $\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$	dM1	1.1b
	So in 2019, **720** students studied **Construction**, **40** students studied **Design** and **350** students studied **Hospitality**	A1	3.2a
		(4)	
			(8 marks)

Notes:
Mark (i) and (ii) together **(a)(i)** **B1**: Defines 3 variables, minimum e.g. construction = C, Design = D, Hospitality = H. This may be seen in text of the question, abbreviations may be used **(ii)**

M1: Finds either the increase or the number of students in 2020. This may be implied by any equation which equals 1113 or 1112.997. If students use 1100 instead of 1110 this is slip and we can award this mark.

M1: Attempts to use the model to set up at least 2 equations

A1: All 3 simplified equations correct (decimals or fractions), one for each different piece of information. Award with mark even if B0 is scored and it is clear what the variables used stand for.

Ignore any additional equations even if incorrect. As soon as 3 correct equations are seen you may award this mark.

Alternative approach

(i) **B1:** Construction = $H + 370$, Design = D, Hospitality = H

(ii) **M1M1A1:** $H + 370 + D + H = 1110$ o.e $C = H + 3701.0125(H + 370) + 1.025D + 0.98H = 1113$ or 1112.997 o.e. they do not need to be simplified

(b) This is M1 M1 A1 A1 on ePen but is marked M1A1M1A1

M1: Uses their equation in part(a) to set up a matrix equation of the form $\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$, where "…" are numerical values.

A1ft: Correct matrix equation for their equations

dM1: Dependent on previous method mark. Writes $\left(\text{their } A()^{-1}\begin{pmatrix} 1110 \\ \text{their "370"} \\ \text{their "3"} \end{pmatrix}\right)$ and obtains at least one value of C, D or H. The inverse matrix need not be found, writing $\mathbf{A}^{-1}\begin{pmatrix} 1110 \\ 370 \\ \text{their "3"} \end{pmatrix} = \ldots$ is sufficient. A correct matrix equation followed by correct values implies this mark.

Condone $\begin{pmatrix} 1110 \\ \text{their "370"} \\ \text{their "3"} \end{pmatrix}\mathbf{A}^{-1} = \ldots$ as long as they reach some values. The values imply the correct method

Note: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{10}{23} = 0.43\ldots & \frac{18}{23} = 0.78\ldots & -\frac{400}{23} = -17.39\ldots \\ \frac{3}{23} = 0.13\ldots & -\frac{13}{23} = -0.56\ldots & \frac{800}{23} = 34.78\ldots \\ \frac{10}{23} = 0.43\ldots & -\frac{5}{23} = -0.21\ldots & -\frac{400}{23} = -17.39\ldots \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{410}{23} = 17.82\ldots & \frac{18}{23} = 0.78\ldots & -\frac{400}{23} = -17.39\ldots \\ -\frac{797}{23} = -34.65\ldots & -\frac{13}{23} = -0.56\ldots & \frac{800}{23} = 34.78\ldots \\ \frac{410}{23} = 17.82\ldots & -\frac{5}{23} = -0.21\ldots & -\frac{400}{23} = -17.39\ldots \end{pmatrix}$$

A1: Interprets the answer in the context of the question, minimum is $C = 720$, $D = 40$, $H = 350$ with their variables. Condone the variable not been defined for this mark if it is clear which variable belong to what course.

Note: they must be using a matrix equation to solve the equation to score any marks.

Alternative approach

For example

Equations simplifies to $C - H = 370$, $D + 2H = 740$ and $1.025D + 1.9925H = 738.375$

which leads to $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1.025 & 1.9925 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 740 \\ 370 \\ 738.375 \end{pmatrix}$ then $\begin{pmatrix} C \\ D \\ H \end{pmatrix} =$

$\begin{pmatrix} 17.826 & 1 & -17.3913 \\ -34.6521 & 0 & 34.7826 \\ 17.826 & 0 & -17.3913 \end{pmatrix}\begin{pmatrix} 740 \\ 370 \\ 738.375 \end{pmatrix} = \begin{pmatrix} 720 \\ 40 \\ 350 \end{pmatrix}$

Note: A 2 x 2 matrix is fine if it is appropriate for their equation.

Special Case: Forming an equation in one variable

(a)(i) B1: Hospitality = x, Construction = $x + 370$, Design = $740 - 2x$

(ii) M1M1A1: $1.0125(x + 370) + 1.025(740 - 2x) + 0.98x = 1113$ or 1112.997

(a)(i) B1: Hospitality = $x - 370$, Construction = x , Design = $1480 - 2x$

(ii) M1M1A1: $1.0125(x) + 1.025(1480 - 2x) + 0.98(x - 370) = 1113$ or 1112.997

(b) M0A0M0A0: They have an equation and are not forming and solving a matrix equation

Q3.

Question	Scheme	Marks	AOs
(a)	a represents the proportion of juvenile chimpanzees that (survive and) **remain** juvenile chimpanzees the next year.	B1	3.4
		(1)	
(b)(i)	Determinant $= 0.82a - 0.08\times 0.15$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \ldots \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \frac{1}{0.82a - 0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	A1	1.1b
		(3)	
(ii)	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \frac{1}{0.82a - 0.012}\begin{pmatrix} 0.82\times 15360 - 0.15\times 43008 \\ (-0.08)\times 15360 + 43008a \end{pmatrix}$ OR forms equations $15360 = aJ_0 + 0.15\times A_0$, $43008 = 0.08\times J_0 + 0.82\times A_0$	M1	3.1a
	$\frac{1}{0.82a - 0.012}\left[6144 + \left(43008a - 1228.8\right)\right] = 64000$ $\Rightarrow 4915.2 + 43008a = 64000(0.82a - 0.012) \Rightarrow a = \ldots$ OR $A_0 = 64000 - J_0 \Rightarrow 43008 = 0.08\times J_0 + 0.82\times\left(64000 - J_0\right) = J_0 = \ldots$ $\Rightarrow a = \frac{15360 - \left(64000 - J_0\right)}{J_0} = \ldots$	M1	3.1a
	$a = \frac{5683.2}{9472} = 0.60$	A1	1.1b
		(3)	
(iii)	Initial juvenile population $= \frac{\text{"6144"}}{\text{"0.48"}} = 12800$	M1	3.4
	So change of 2560 juvenile chimpanzees	A1	1.1b
		(2)	
(c)	As the number of juveniles has increased, the model is not initially predicting a decline, so is not suitable in the short term. (Follow through their answer to (b) – but they must have made an attempt at it to find at least a value for J_0)	B1ft	3.5a
		(1)	
(d)	Third category needs to be introduced for chimpanzees aged 40 and above, mature chimpanzees M_n, and a matrix multiplication of increased dimension set up. Accept $3\times 3, 3\times 2$ or 2×3 matrices including all three categories in the column vector.	M1	3.5c
	The corresponding matrix model will have the form $\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & \underline{0} \\ 0.08 & c & 0 \\ 0 & d & e \end{pmatrix}\begin{pmatrix} J_n \\ A_n \\ M_n \end{pmatrix}$ (The underlined zero must be correct but do not be concerned about any values used in the other entries.)	A1	3.3
		(2)	
		(12 marks)	

Notes		
(a)	**B1**	Correct interpretation. Need not mention survival but must be clear it is the (proportion of) **juveniles that remain as juveniles** the next year (ie those that survive but don't progress to adulthood). E.g. accept "(number of) juveniles who do not become adults" but do not accept "surviving juveniles".
		Mark part (b) as a whole.
(b)(i)	**M1**	Attempts the determinant in terms of a Allow miscopies for the attempt. Allow $0.82a - 0.12$ as a slip.
	M1	Attempts the form of the inverse, swapped leading diagonals and sign changed on both off diagonals. Allow miscopies of the numbers but the signs must be correct.
	A1	Correct inverse matrix
(ii)	**M1**	Use the inverse matrix and attempts to find the initial juvenile and adult populations. . (May have determinant 1 for this mark.) Alternatively, sets up simultaneous equations from the original system, $15360 = aJ_0 + 0.15 \times A_0$ and $43008 = 0.08 \times J_0 + 0.82 \times A_0$ Accept with J_n and A_n or other appropriate variables.
	M1	Uses the sum of initial populations equals 64000 in an attempt to find a. (May have determinant 1 for this mark.) If using alternative, use of e.g. $A_0 = 64000 - J_0$ in second equation to find J_0, followed by attempt to find a. Award for an attempt to solve the equations, but don't be too concerned with the algebraic process as long as they are attempting to use all three equations.
	A1	Correct value, $a = 0.6$ (or 0.60 or $\frac{3}{5}$).
(iii)	**M1**	Uses their a to find the value of J_0. This mark may be gained for work done in (ii) if the alternative has been used but must have come from a correct method.
	A1	Correct difference found, as long as there is no contradictory statement – so "decrease of 2560" is A0.
(c)	**B1ft**	Comments that the change is an increase so does not fit the model. Follow through their answer to (b) as long as at least a value for J_0 has been found. If a decrease has been found allow for commenting the model is suitable. If an answer is given to (b)(iii), follow through on whatever their answer is. If no answer has been given, but an initial population found, a comparison should be made between this value and 153600 with conclusion must be consistent with their answer for J_0
(d)	**M1**	Introduces a third category (may be *M*ature, *E*lderly or any suitable letter used) and sets up a matrix multiplication (the left hand side may be missing for this mark) with all three categories in the column vector. The dimension of the matrix should be 3 in at least either row or column, and there should be a 3×1 vector.
	A1	Sets up the new matrix equation, including both sides and making clear the zero (underlined) so that the correct progression that no new juveniles arise from the mature chimpanzees is clear. Overlook other values, though ideally the other two zeroes are shown too, to indicate mature chimpanzees do not regress to adulthood, and juveniles cannot proceed directly to mature chimpanzees.

Appendix: Alternatives to (b)

Note that variations may occur with the line equation chosen in part (a), but mark as follows:

Question	Scheme	Marks	AOs
Alt 1 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $=380000\left(\lambda - \frac{3}{4}\right)^2 + 48750 \Rightarrow \lambda = ...$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{"48750"} = ...$	M1	1.1b
	Awrt 221m from correct working, so completion of square must have been correct. (Must include units)	A1	1.1b
		(2)	

Notes

(i)	**B1M1**	As per main scheme.
	M1	Realises the need to find the distance from the point on the mountain to a general point on the line.
	dM1	Attempts the distance or distance squared of $\overline{MX}$, expands and completes the square to find the value of λ for which distance is minimum. May obtain other forms for the completed square. Look for $A(B\lambda - C)^2 - D + "262500"$ where $A, B, C, D \neq 0$ but B may be 1.
	M1A1	As per main scheme.
(ii)	**M1**	Correct method for the distance. May be as per main scheme, or via extracting from the completed square constant term.
	A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m

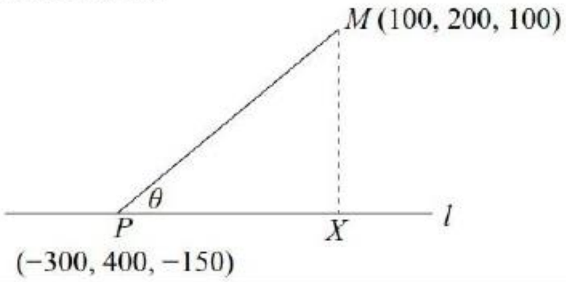
Alt 2 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $\frac{d}{dx}(d^2) = 0 \Rightarrow 760000\lambda - 570000 = 0 \Rightarrow \lambda = ...$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{(150-100)^2 + (325-200)^2 + (-75-100)^2} = ...$	M1	1.1b
	Awrt 221m from correct working, differentiation etc must have been correct. (Must include units)	A1	1.1b
		(2)	

Notes		
		As per main scheme except for:
(i)	dM1	Attempts the distance or distance squared of $\overrightarrow{MX}$, differentiates and set to zero to find λ for minimum distance.
(ii)	M1	May substitute λ into the distance squared formula to find distance.

Alt 3	$k = 200$	B1	2.2a
(b)(i)	If M is the point on mountain, then e.g (may use Q rather than P) $\overrightarrow{MP} = \begin{pmatrix}-400\\200\\-250\end{pmatrix} \Rightarrow \cos\theta = \dfrac{\begin{pmatrix}-400\\200\\-250\end{pmatrix}.\begin{pmatrix}600\\-100\\100\end{pmatrix}}{\sqrt{(-400)^2+200^2+(-250)^2}\sqrt{600^2+(-100)^2+100^2}}$ $\Rightarrow \cos\theta = \ldots$ or $\theta = \ldots$ (where θ is the angle between the line and $\overrightarrow{MP}$)	M1	3.1b
	$\Rightarrow \lvert\overrightarrow{PX}\rvert = \lvert\overrightarrow{MP}\rvert\cos\theta = \ldots$	dM1	1.1b
	So e.g. $\overrightarrow{OX} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \dfrac{\lvert\overrightarrow{PX}\rvert}{\left\lvert\begin{pmatrix}600\\-100\\100\end{pmatrix}\right\rvert}\begin{pmatrix}600\\-100\\100\end{pmatrix} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \dfrac{"75\sqrt{8}"}{100\sqrt{38}}\begin{pmatrix}600\\-100\\100\end{pmatrix} = \ldots$	M1	3.4
	So coordinates of X are (150, 325, −75) Accept as $\begin{pmatrix}150\\325\\-75\end{pmatrix}$	A1	1.1b
		(5)	
(ii)	Length of tunnel is $\lvert\overrightarrow{MP}\rvert\sin\theta = \ldots$ (oe)	M1	1.1b
	Awrt 221m from correct working. (Must include units)	A1	1.1b
		(2)	

Notes		
(i)	B1	Correct value of k deduced.
	M1	Finds $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$) and attempts scalar product formula with this and the direction of the line to find the angle or cosine of the angle between line and $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$)
	dM1	Uses their angle with the cosine to find the length of $\overrightarrow{PX}$ (or $\overrightarrow{QX}$). Accept equivalent trigonometric methods (e.g. finding opposite side first and using tangent or Pythagoras.
	M1	Uses the length of and $\overrightarrow{PX}$ (or $\overrightarrow{QX}$) to find the coordinates of the point on the line at shortest distance from M.
	A1	Correct point.
(ii)	M1	Correct method for the distance. May be as per main scheme, or use of sine ratio with their angle between the line and and $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$). Accept equivalent trigonometric methods.
	A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m

Useful diagram:



Note for P, $\cos\theta = \pm\dfrac{57}{\sqrt{38}\sqrt{105}}$,

$\theta = 25.5\ldots°$ and $\lvert\overrightarrow{PX}\rvert = 75\sqrt{38}$

For Q $\cos\theta = \pm\dfrac{19}{\sqrt{38}\sqrt{29}}$,

$\theta = 55.08\ldots°$, $\lvert\overrightarrow{QX}\rvert = 25\sqrt{38}$

Q4.

Question	Scheme	Marks	AOs
(a)	$det(\boldsymbol{M}) = a(6) - 2(4) - 3(2a - 12)$	M1	1.1b
	$det(\boldsymbol{M}) = 28 \neq 0$ therefore, non-singular for all values of a	A1	2.4
		(2)	
(b)	Finds the matrix of minors $\begin{pmatrix} 6 & 4 & 2a-12 \\ 4+3a & 2a+12 & a^2-8 \\ 9 & 6 & 3a-4 \end{pmatrix}$	M1	1.1b
	Finds the matrix of cofactors and transposes. $\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1	1.1b
	$\frac{1}{28}\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1 A1	1.1b 2.1
		(4)	
			(6 marks)

Notes:

(a)

M1: Finds the determinant of the matrix **M**. Must be seen in part (a). Allow one slip if no method shown.

A1: Correct value for determinant, states doesn't equal 0 (accept > 0) and draws the conclusion that the matrix is non-singular. If non-singular meaning determinant is non-zero is given in a preamble then accept a minimal conclusion (e.g. "hence shown"), but there must be a conclusion.

(b)

M1: Finds the matrix of minors, at least 5 correct values.

M1: Finds the matrix of cofactors and transposes (in either order). Note: some will do all these steps in one go, which is fine as long as it is clear what they have done. Allow minor slips if the process is clearly correct.

M1: Completes the process to find the inverse matrix, divides by the determinant.

A1: Correct matrix.

Q5.

Question	Scheme	Marks	AOs
(a)	$\mathbf{MN}=\begin{pmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & -10k+50 \\ 4k-20 & 0 & -14 \end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(b)(i)	$\mathbf{MN}=\begin{pmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{pmatrix}$	B1ft	1.1b
(ii)	$\mathbf{M}^{-1}=-\frac{1}{14}\begin{pmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$	B1	1.1b
		(2)	
(c)	$\mathbf{M}^{-1}=-\frac{1}{14}\begin{pmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}=\ldots$	M1	1.1b
	$\left(-\frac{12}{7}, \frac{40}{7}, -\frac{1}{14}\right)$	A1	1.1b
		(2)	
(d)	The coordinates of the only point at which the **planes** represented by the equations in (c) meet.	B1	2.2a
		(1)	
		(7 marks)	

Notes
(a) B1: For 2 correct rows or 2 correct columns (allow unsimplified) B1: Fully correct simplified matrix (b)(i) B1ft: Correct matrix (follow through from part (a)). If an error with part (a) allow the correct matrix stated, restart use of calculator. (ii) B1: Deduces the correct inverse matrix, may use calculator (c) M1: Any complete method to find the values of x, y and z (Must be using **their inverse** if using the method in the main scheme) Allow use of a calculator A1: Correct exact coordinates (allow as a vector or $x = \ldots, y = \ldots, z = \ldots$) (d) B1: Describes the correct geometrical configuration of the **planes**

Q6.

Question	Scheme	Marks	AOs
(a)	(i) $\mathbf{AB}=\begin{pmatrix}4&-1\\7&2\\-5&8\end{pmatrix}\begin{pmatrix}2&3&2\\-1&6&5\end{pmatrix}=\begin{pmatrix}8+1&12-6&8-5\\14-2&21+12&14+10\\-10-8&-15+48&-10+40\end{pmatrix}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}$	M1	1.1b
	So $\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}-\begin{pmatrix}-15&6&3\\12&9&24\\-18&33&6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$ or $\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}+\begin{pmatrix}15&-6&-3\\-12&-9&-24\\18&-33&-6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$ and states a value for k	M1	1.1b
	Hence $\mathbf{AB}-3\mathbf{C}-24\mathbf{I}=\mathbf{0}$ so $k=-24$	A1	1.1b
	(ii) Need two things One of: • **BA** is a 2×2 matrix • Finds the matrix **BA** (must be a 2×2 matrix) AND One of: • cannot subtract a 3×3 matrix • finds matrix **3C** and comments that they have different dimensions / can't be done • can't subtract matrices of different sizes • **3C** or **C** is a 3×3 matrix • **BA** needs to be a 3×3 matrix	B1	2.4
		(4)	

(b)(i)	$\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-14\\3\\7\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$ Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{C}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$ Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}$	M1	1.2
(ii)	$=\frac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}=\ldots$	M1	1.1b

	$=\begin{pmatrix} -\frac{41}{180} & \frac{7}{360} & \frac{13}{360} \\ -\frac{7}{45} & -\frac{1}{90} & \frac{11}{90} \\ \frac{31}{180} & \frac{43}{360} & -\frac{23}{360} \end{pmatrix}\begin{pmatrix} -14 \\ 3 \\ 7 \end{pmatrix}=\ldots$ $\mathbf{C}^{-1}\begin{pmatrix} -14 \\ 3 \\ 7 \end{pmatrix}=\ldots$		
	So solution is $x=\frac{7}{2}, y=3, z=-\frac{5}{2}$ or $(3.5,\ 3,\ -2.5)$	A1	1.1b
		(3)	
			(7 marks)

Notes:

(a) (i)

M1: Attempts to find **AB**. Usually this will be done on calculator so answer implies the method. If answer is incorrect allow for at least 6 correct entries or calculations shown.
This mark can be implied by a correct matrix for $\mathbf{AB}-3\mathbf{C}$ gives the first M1

M1: Uses their **AB** and **3C** matrices to find a multiple **I** and states a value for k

A1: Correct proof with $k=-24$ seen explicitly (may be in equation).

Minimum working required is $\mathbf{AB}-3\mathbf{C}=\begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ gets M1 then states a value for k M1
then $k=-24$ gets A1

Special case: If minimum working required is not seen and just $k=-24$ stated then M1 M0 A0 as they have not shown that the value of k works.

(ii)

B1: Correct explanation referring to the dimensions of **BA** and **C** (or 3**C**) and that they do not match in the equation. They can find both these matrices and then comment they cannot be subtracted.

(b) Mark (i) and (ii) altogether

M1: States or implies use of the correct method of using the inverse matrix.

M1: Carries out the process of multiplying after finding the inverse. May find inverse long hand first. Finding the inverse matrix then writes down an answer gains M1.

Note: There is no need to find the inverse matrix. If the inverse matrix is not stated just answers written down then two out of the three correct ordinates imply the M1.

A1: Correct solution. Must be clear that $x=\frac{7}{2}, y=3, z=-\frac{5}{2}$ allow $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3.5 \\ 3 \\ -2.5 \end{pmatrix}$

Note: If they solve using simultaneous equations only this is M0 M0 A0
If there is no reference to the inverse matrix and correct answers stated this is M0 M0 A0

Q7.

Question	Scheme	Marks	AOs
(a)	$\lvert\mathbf{M}\rvert = 2(-k-8)+1(-3-12)+1(6-3k)=0 \Rightarrow k=...$	M1	1.1b
	$k \neq -5$	A1	2.4
		(2)	
(b) Way 1	$\mathbf{M}=\begin{pmatrix}2 & -1 & 1\\ 3 & -6 & 4\\ 3 & 2 & -1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix}$	M1	3.1a
	$\mathbf{M}^{-1}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}$	B1	1.1b
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=...$	M1	2.1
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2p+1\\ 15p-5\\ 24p-7\end{pmatrix}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5
		(5)	
(b) Way 2	$\begin{aligned}2x - y + z &= p\\ 3x-6y+4z &= 1\\ 3x+2y-z &= 0\end{aligned} \Rightarrow$ e.g. $\begin{aligned}8y-5z&=-1\\ 9y-5z&=3p-2\end{aligned} \Rightarrow y=...$ $\Rightarrow x=..., z=...$	M1	3.1a
	$y=3p-1$(or $x=\frac{-2p+1}{5}$ or $z=\frac{24p-7}{5}$)	B1	1.1b
	$8(3p-1)-5z=-1 \Rightarrow z=... \Rightarrow x=...$	M1	2.1
	$z=\frac{24p-7}{5},\ x=\frac{-2p+1}{5}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5

(c)(i)	For consistency: E.g. $5x+y=4-q$ and $15x+3y=q$	M1	3.1a
	$4-q=\frac{q}{3} \Rightarrow q=...$	M1	2.1
	$q=3$	A1	1.1b
	Alternative for (c)(i): $x=1 \Rightarrow 2-y+z=1,\ 3+2y-z=0 \Rightarrow y=...,z=...$ M1 for allocating a number to one variable and solves for the other 2 $x=1, y=-4, z=-5 \Rightarrow 3+20-20=q$ M1 substitutes into the second equation and solves for q A1: $q=3$		
(ii)	Three **planes** that intersect in a **line** Or Three **planes** that form a **sheaf** allow **sheath**!	B1	2.4
		(4)	
			(11 marks)

Notes
(a) M1: Attempts determinant, equates to zero and attempts to solve for k in order to establish the restriction for k. For the determinant, at least 2 of the 3 "elements" should be correct. May see rule of Sarrus used for determinant e.g. $\lvert\mathbf{M}\rvert = (2)(k)(-1)+(4)(3)(-1)+(3)(2)(1)-(3)(k)(-1)-(2)(4)(2)-(-1)(3)(-1)=0 \Rightarrow k=...$ A1: Describes the correct condition for k with no contradictions. Allow e.g. $k < -5$, $k > -5$ (b)**Way 1** M1: A complete strategy for solving the given equations. Need to see an attempt at the inverse followed by a correct method for finding x, y and z B1: Correct inverse matrix M1: Uses their inverse and attempts the multiplication with the correct vector A1: Correct values for x, y and z in any form A1ft: Correct values given in coordinate form only. **Follow through their x, y and z.** **Way 2** M1: A complete strategy for solving the given equations. Need to see an attempt at eliminating one variable followed by a correct method for finding x, y and z B1: One correct value M1: Uses the equations to find values for the other 2 variables A1: Correct values for x, y and z in any form A1ft: Correct values given in coordinate form only. **Follow through their x, y and z.** (c)(i) M1: Uses a correct strategy that will lead to establishing a value for q. E.g. eliminating one of x, y or z M1: Solves a suitable equation to obtain a value for q A1: Correct value (ii) B1: Describes the correct geometrical configuration. Must include the **two** ideas of **planes** and meeting in a **line** or forming a **sheaf** with no contradictory statements.

Q8.

Question	Scheme	Marks	AOs
(i) (a)	It is possible as the number of columns of matrix **A** matches the number of rows of matrix **B**.	B1	2.4
(b)	It is not possible as matrix **A** and matrix **B** have different dimensions o.e. different number of columns	B1	2.4
		(2)	
(ii) (a)	$\lambda = 5$	B1	2.2a
	$a = 1,\ b = 2$	B1	2.2a
(b)	Inverse matrix $= \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$	B1 ft	3.1a
		(3)	
(iii)	A complete method to find the determinant of the matrix and set equal to zero.	M1	1.1b
	Determinant = $1(\sin\theta\sin 2\theta - \cos\theta\cos 2\theta) - 1(0) + 1(0) = 0$	A1	1.1b
	Uses compound angle formula to achieve $\cos 3\theta = 0$ leading to $\theta = ...$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 1 - 2\sin^2 q$ (e.g. to achieve $\cos q(4\sin^2 q - 1) = 0$) leading to $\theta = ...$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 2\cos^2 q - 1$ (e.g. to achieve $4\cos^3 q - 3\cos q = 0$) leading to $\theta = ...$	M1	3.1a
	$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$	A1	1.1b
		(4)	
		(9 marks)	

Notes:
(i)(a) **B1**: Comments that the number of columns of matrix **A** (2) equals the number of rows of matrix **B** (2) therefore it is possible. Accept other terminology that is clear in intent e.g. "length of **A**" and "height of **B**" **(b)** **B1**: Comments that matrix **A** and matrix **B** have different dimensions therefore it is not possible.
(ii)(a) **B1**: Deduces the correct value for $\lambda = 5$ **B1**: Deduces the correct values for a and b **(b)** **B1ft**: Identifies and applies a correct method find the inverse matrix. May multiply from the given equation, in which case follow through on their value of lambda. Alternatively, award for a correct matrix found by calculator or long hand having found a and b and using these values in the matrix.

(iii) **M1**: A complete method to find the determinant of the matrix and sets it equal to 0 **A1**: Correct equation **M1**: Uses appropriate correct trig identities to solve the equation and finds a value for q **A1**: All three correct values $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and no others in the range.