Integration Techniques

Questions

Q1.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n 2x \, dx$$

(a) Prove that for $n \geq 2$

$$I_n = \frac{n-1}{n} I_{n-2}$$

(4)

(b) Hence determine the exact value of

$$\int_0^{\frac{\pi}{2}} 64 \sin^5 x \cos^5 x \, dx$$

(3)

(Total for question = 7 marks)

Q2.

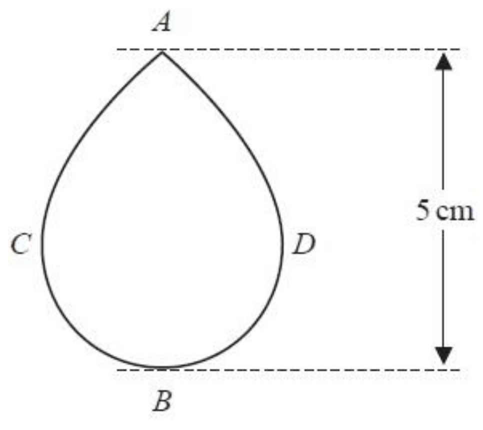

Figure 1

Figure 1 shows the vertical cross section of a child's spinning top. The point *A* is vertically above the point *B* and the height of the spinning top is 5 cm.

The line *CD* is perpendicular to *AB* such that *CD* is the maximum width of the spinning top.

The spinning top is modelled as the solid of revolution created when part of the curve with polar equation

$$r^2 = 25 \cos 2\theta$$

is rotated through 2π radians about the initial line.

(a) Show that, according to the model, the surface area of the spinning top is

$$k\pi(2 - \sqrt{2})\,\text{cm}^2$$

where k is a constant to be determined.

(7)

(b) Show that, according to the model, the length CD is $\dfrac{5\sqrt{2}}{2}$ cm.

(6)

(Total for question = 13 marks)

Q3.

$$I_n = \int \operatorname{cosec}^n x \,dx \qquad n \in \mathbb{Z}$$

(a) Prove that, for $n \geq 2$

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$$

(4)

(b) Hence show that

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \,dx = \frac{56}{135}\sqrt{3}$$

(4)

(Total for question = 8 marks)

Q4.

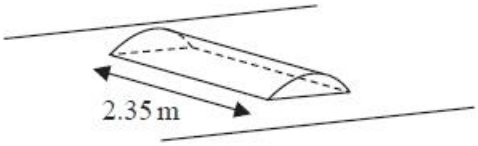

Figure 1

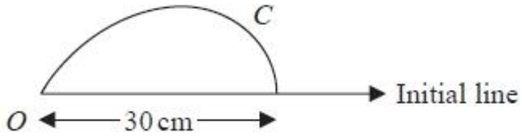

Figure 2

Figure 1 shows a sketch of a design for a road speed bump of width 2.35 metres. The speed bump has a uniform cross-section with vertical ends and its length is 30 cm. A side profile of the speed bump is shown in Figure 2.

The curve *C* shown in Figure 2 is modelled by the polar equation

$$r = 30(1 - \theta^2) \qquad 0 \leqslant \theta \leqslant 1$$

The units for *r* are centimetres and the initial line lies along the road surface, which is assumed to be horizontal.

Once the speed bump has been fixed to the road, the visible surfaces of the speed bump are to be painted.

Determine, in cm^2, the area that is to be painted, according to the model.

(Total for question = 10 marks)

Q5.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geq 2$,

$$nI_n = (n-1)I_{n-2}$$

(4)

(b)

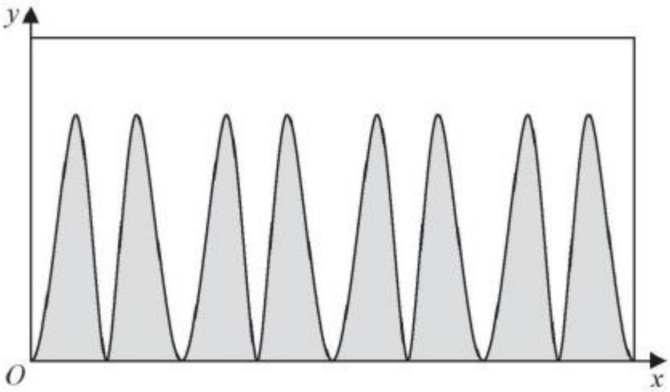

Figure 2

A designer is asked to produce a poster to completely cover the curved surface area of a solid cylinder which has diameter 1m and height 0.7 m.

He uses a large sheet of paper with height 0.7 m and width of π m.

Figure 2 shows the first stage of the design, where the poster is divided into two sections by a curve.

The curve is given by the equation

$$y = \sin^2 (4x) - \sin^{10} (4x)$$

relative to axes taken along the bottom and left hand edge of the paper.

The region of the poster below the curve is shaded and the region above the curve remains unshaded, as shown in Figure 2.

Find the exact area of the poster which is shaded.

(5)

(Total for question = 9 marks)

Q6.

The curve *C* has equation

$$y = \ln\left(\frac{e^x + 1}{e^x - 1}\right), \qquad \ln 2 \leqslant x \leqslant \ln 3$$

(a) Show that

$$\frac{dy}{dx} = -\frac{2e^x}{e^{2x} - 1}$$

(4)

(b) Find the length of the curve *C*, giving your answer in the form ln *a*, where *a* is a rational number.

(6)

(Total for question = 10 marks)

Q7.

$$I_n = \int_0^{\ln 2} \cosh^n x \, dx, \quad n \geqslant 0$$

(a) Show that, for $n \geq 2$,

$$I_n = \frac{3a^{n-1}}{nb^n} + \frac{n-1}{n} I_{n-2}$$

where *a* and *b* are integers to be found.

(6)

(b) Hence, or otherwise, find the exact value of

$$\int_0^{\ln 2} \cosh^4 x \, dx$$

(4)

(Total for question = 10 marks)

Q8.

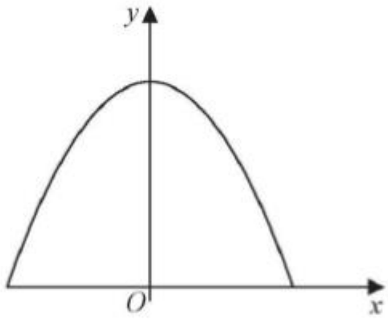

Figure 1

An engineering student makes a miniature arch as part of the design for a piece of coursework.

The cross-section of this arch is modelled by the curve with equation

$$y = A - \tfrac{1}{2}\cosh 2x, \quad -\ln a \leqslant x \leqslant \ln a$$

where $a > 1$ and A is a positive constant. The curve begins and ends on the x-axis, as shown in Figure 1.

(a) Show that the length of this curve is $k\left(a^2 - \frac{1}{a^2}\right)$, stating the value of the constant k.

(5)

The length of the curved cross-section of the miniature arch is required to be 2 m long.

(b) Find the height of the arch, according to this model, giving your answer to 2 significant figures.

(4)

(c) Find also the width of the base of the arch giving your answer to 2 significant figures.

(1)

(d) Give the equation of another curve that could be used as a suitable model for the cross-section of an arch, with approximately the same height and width as you found using the first model.
(You do not need to consider the arc length of your curve)

(2)

(Total for question = 12 marks)

Q9.

In this question you must show all stages of your working.

You must not use the integration facility on your calculator.

$$I_n = \int t^n \sqrt{4 + 5t^2}\, \mathrm{d}t \qquad n \geqslant 0$$

(a) Show that, for $n > 1$

$$I_n = \frac{t^{n-1}}{5(n+2)}\left(4 + 5t^2\right)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)} I_{n-2}$$

(5)

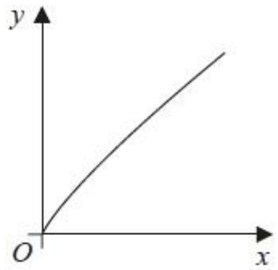

Figure 1

The curve shown in Figure 1 is defined by the parametric equations

$$x = \frac{1}{\sqrt{5}} t^5 \qquad y = \frac{1}{2} t^4 \qquad 0 \leqslant t \leqslant 1$$

This curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

(5)

$$\pi \int_0^1 t^7 \sqrt{4 + 5t^2}\, \mathrm{d}t$$

Using the results in parts (a) and (b) and making each step of your working clear,

(c) determine the value of the external surface area of the shell, giving your answer to 3 significant figures.

(5)

(Total for question = 15 marks)

Mark Scheme – Integration Techniques

Q1.

Question	Scheme	Marks	AOs
(a)	$I_n = \int \sin^n 2x\,dx = \int \sin^{n-1} 2x \sin 2x\,dx$ Leading to $I_n = \left[\lambda \sin^{n-1} 2x \cos 2x\right] - \mu \int \sin^{n-2} 2x \cos^2 2x dx$	M1	2.1
	$I_n = \left[-\frac{1}{2}\sin^{n-1} 2x \cos 2x\right] + \int (n-1)\sin^{n-2} 2x \cos^2 2x dx$	A1	1.1b
	$I_n = 0 + (n-1)\int \sin^{n-2} 2x\left(1-\sin^2 2x\right)dx$ $= (n-1)\int \sin^{n-2} 2x\,dx - (n-1)\int \sin^n 2x\,dx$	dM1	1.1b
	$nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{n-1}{n}I_{n-2}$ *	A1*	2.1
		(4)	
	Alternative: On epen MAMA make sure marks are recorded in the correct place $I_n = \int \sin^n 2x\,dx = \int \sin^{n-2} 2x \sin^2 2x\,dx$ $= \int \sin^{n-2} 2x\,(1-\cos^2 2x)\,dx$ $= \int \sin^{n-2} 2x\,dx - \int \sin^{n-2} 2x \cos^2 2x\,dx$ Leading to an attempt at integration	M1	1.1b
	$I_n = I_{n-2} - \int (\sin^{n-2} 2x \cos 2x)(\cos 2x)\,dx$ $I_n = I_{n-2} - \left[\lambda \sin^{n-1} 2x \cos 2x\right] + \mu \int \sin 2x \sin^{n-1} 2x dx$	dM1	2.1
	$I_n = I_{n-2} - \left[\frac{1}{2(n-1)}\sin^{n-1} 2x \cos 2x\right] - \frac{1}{n-1}\int \sin^n 2x dx$	A1	1.1b
	$I_n = I_{n-2} - \frac{1}{n-1}I_n \Rightarrow (n-1)I_n = (n-1)I_{n-2} - I_n \Rightarrow I_n = \frac{n-1}{n}I_{n-2}$*	A1*	2.1
(b)	$\int 64\sin^5 x \cos^5 x\,dx = \int A\sin^5 2x\,dx$ Note $A = 2$	M1	2.1
	$I_5 = \frac{4}{5}I_3,\ I_3 = \frac{2}{3}I_1$ and $I_1 = \int_0^{\frac{\pi}{2}} \sin 2x\ dx = [\alpha \cos 2x]_0^{\frac{\pi}{2}} = \ldots$	M1	1.1b
	$= 2 \times \left(\frac{4}{5}\right) \times \left(\frac{2}{3}\right) \times 1 = \frac{16}{15}$	A1	1.1b
		(3)	
			(7 marks)

Notes:

(a)

M1: Writes $sin^n 2x$ as $sin^{n-1} 2x \ \ sin 2x$ and integrates using by parts to the form

$I_n = \left[\lambda \sin^{n-1} 2x \cos 2x\right] - \mu \int \sin^{n-2} 2x \cos^2 2x \, dx$

A1: Correct integration, may be unsimplified

dM1: Substitutes the limits of 0 and $\frac{\pi}{2}$ into 'uv', this may be implied by 0. Replaces $cos^2 2x = 1 - sin^2 2x$ and multiplies out into separate integrals.

A1*: Achieves the printed answer following a correct intermediate line and no errors. Cso

Alternative

M1: Writes $sin^n 2x$ as $sin^{n-2} 2x \ \ sin^2 2x = sin^{n-2} 2x \ (1 - cos^2 2x)$, writes as

$= \int \sin^{n-2} 2x \ dx - \int \sin^{n-2} 2x \cos^2 2x \, dx$ and attempts to integrate

dM1: Integrates using by parts to the form $I_n = I_{n-2} - \left[\lambda \sin^{n-1} 2x \cos 2x\right] + \mu \int \sin 2x \sin^{n-1} 2x dx$

A1: Correct integration, may be unsimplified

A1*: Achieves the printed answer following a correct intermediate line and no errors. cso

(b)

M1: Uses the identity $\boldsymbol{sin}\, 2x = 2\, \boldsymbol{sin}\, x\, \boldsymbol{cos}\, x$ in an attempt to write the integral as $\int A\, \boldsymbol{sin}^5\, 2x \ \boldsymbol{dx}$

M1: Uses the answer to part (a) to find a value for I_5 and I_3 and finds $I_1 = \int_0^{\frac{\pi}{2}} sin\, 2x \ \ dx = \ldots$

A1: $\frac{16}{15}$ cso

Q2.

Question	Scheme	Marks	AOs
	$\text{SA} = 2\pi\int r\sin\theta\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = 2\pi\int r\sin\theta\sqrt{25\cos 2\theta+...}\,\mathrm{d}\theta$ Or $\text{SA} = 2\pi\int r\cos\theta\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = 2\pi\int r\cos\theta\sqrt{25\cos 2\theta+...}\,\mathrm{d}\theta$	M1	2.1
	$r^2 = 25\cos 2\theta \Rightarrow 2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = k\sin 2\theta$ Or $r = 5\cos^{\frac{1}{2}} 2\theta \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} = A\cos^{-\frac{1}{2}} 2\theta \times B\sin 2\theta$ (oe)	M1	2.1
	$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -50\sin 2\theta$ or $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{-50\sin 2\theta}{2r}$ Or $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{5}{2}\cos^{-\frac{1}{2}} 2\theta \times -2\sin 2\theta$ (oe)	A1	1.1b
	$\text{SA} = 2\pi\int 5\sqrt{\cos 2\theta}\sin\theta\sqrt{25\cos 2\theta + \frac{25\sin^2 2\theta}{\cos 2\theta}}\,\mathrm{d}\theta$ $= 2\pi\int 5\sqrt{\cos 2\theta}\sin\theta\frac{5}{\sqrt{\cos 2\theta}}\,\mathrm{d}\theta = k\pi\int\sin\theta\,\mathrm{d}\theta$ (may use cos θ)	M1	2.1
	$= 50\pi\int\sin\theta\,\mathrm{d}\theta$ or $50\pi\int\cos\theta\,\mathrm{d}\theta$	A1	1.1b
	$= 50\pi\int_0^{\frac{\pi}{4}}\sin\theta\,\mathrm{d}\theta = 50\pi\left[-\cos\theta\right]_0^{\frac{\pi}{4}}$	M1	3.4
	$= 25\pi\left(2-\sqrt{2}\right)$ (cm²)	A1	2.2a
		(7)	

(b)	Adopts the correct strategy by: Attempting $\frac{\mathrm{d}y}{\mathrm{d}\theta}$, finding θ when $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ and using their value of θ to find CD	M1	3.1a
	$y = r\sin\theta = 5\sqrt{\cos 2\theta}\sin\theta$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{5\sin 2\theta\sin\theta}{\sqrt{\cos 2\theta}} + 5\sqrt{\cos 2\theta}\cos\theta$	M1	1.1b
	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 5\cos\theta - 20\cos\theta\sin^2\theta = 0 \Rightarrow \theta = ...$	M1	2.1
	E.g. $\sin^2\theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{6}$ or $\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$	A1	1.1b
	$CD = 2r\sin\frac{\pi}{6} = 2\times 5\times\sqrt{\cos\frac{\pi}{3}}\times\frac{1}{2}$; $= \frac{5\sqrt{2}}{2}$ (cm)*	M1; A1*	3.4 2.1
		(6)	
			(13 marks)

Notes

(a)

M1: Applies the surface area formula about the x or y axis with substitution of at least the r^2 and attempt at $\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$ as shown in scheme. May be completed in stages, so allow if correct formula quoted and the relevant "pieces" are found. The 2π may be recovered later but must be present at some stage.

M1: Attempts to find an expression in $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ via implicit differentiation or first square rooting and then using the chain rule.

A1: Correct expression for or in $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ need not be simplified.

M1: Makes a complete substitution into the SA formula and applies appropriate trigonometric identities to simplify to the form $k\pi\int \sin\theta \,\mathrm{d}\theta$ or $k\pi\int \cos\theta \,\mathrm{d}\theta$ as appropriate for their method.

A1: Obtains a correct simplified integral.

M1: Uses the model with appropriate limits to determine the surface area of the top using their integral.

Note that for rotation around the x axis appropriate limits will likely be 0 and $\frac{\pi}{4}$ but may be $-\frac{\pi}{4}$ and 0.

For rotation about the y axis allow this mark for limits $\frac{\pi}{4}$ and $\frac{\pi}{2}$ (note that the curve is not strictly defined for these limits).

A1: Correct expression with no errors. Must come from a correct integral – so if rotation about the y axis is used they must have made clear reference to using $r^2 = -25\cos 2\theta$ as the curve.

(b)

M1: A complete method for finding CD. Need to see the maximum identified and the length of CD calculated (watch out as r is the same value as CD so check they are finding CD)

M1: Uses the product rule correctly to differentiate $r\sin\theta$ – expect the correct form

$$\alpha\sin 2\theta\sin\theta(\cos 2\theta)^{-\frac{1}{2}} + \beta\cos\theta(\cos 2\theta)^{\frac{1}{2}}$$

M1: Makes progress by setting their derivative (which may be $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta}$) equal to 0 and proceeding via correct trig work to reach a value for θ. Various routes are possible e.g.

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 5\cos\theta - 20\cos\theta\sin^2\theta = 0 \Rightarrow \cos\theta(1-4\sin^2\theta) = 0 \Rightarrow \sin\theta = ... \Rightarrow \theta = ...$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = ... \Rightarrow \theta = ...$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow \tan 2\theta = \frac{1}{\tan\theta} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{\tan\theta} \Rightarrow \tan^2\theta = ... \Rightarrow \theta = ...$$

A1: Correct value for θ from correct working – derivative must have been correct. May be implied by correct sin and cosine values used in formulae. SC Award for $\frac{\pi}{3}$ if using $x = r\cos\theta$

M1: Uses their value of θ in the model to find CD, ie $CD = 2\times 5\sqrt{\cos\text{"}2\theta\text{"}}\times\sin\text{"}\theta\text{"}$

Allow for use of $2r\cos\theta$ for attempts stemming from $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$

A1*:cso Correct proof

NB for $x = r\cos\theta$ used, a maximum M0M0M1A1M1A0 can be gained unless $r^2 = -25\cos 2\theta$ is used, in which case full marks is possible.

ALT for (b): M1: for correct overall strategy of finding expression for width and maximising (via any valid method, e.g completion of square, calculus). M1: $CD = 2r\sin\theta = 10\sqrt{\cos 2\theta \sin^2\theta} = 10\sqrt{\sin^2\theta - 2\sin^4\theta}$ forms trig expression inside square root. M1A1: $\sin^2\theta - 2\sin^4\theta = -2\left(\sin^4\theta - \frac{1}{2}\sin^2\theta\right) = -2\left(\left(\sin^2\theta - \frac{1}{4}\right)^2 - \frac{1}{16}\right)$ completes the square. Alternatively may optimise via calculus – score for full method leading to a value for θ M1: Hence max value for CD is $10\times\sqrt{"\frac{1}{8}"}$. Correct method to achieved CD. A1: Correct proof.

Q3.

Question	Scheme	Marks	AOs
(a)	$I_n = \int \text{cosec}^{n-2}x\,\text{cosec}^2 x\,dx$ $u = \text{cosec}^{n-2}x, \frac{dv}{dx} = \text{cosec}^2 x, \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$	M1	2.1
	$I_n = -\text{cosec}^{n-2}x\cot x - (n-2)\int \text{cosec}^{n-3}x(-\text{cosec}x\cot x)(-\cot x)\,dx$	A1	1.1b
	$I_n = -\text{cosec}^{n-2}x\cot x - (n-2)\int \text{cosec}^{n-2}x\cot^2 x\,dx$		
	$I_n = -\text{cosec}^{n-2}x\cot x - (n-2)\int \text{cosec}^{n-2}x(\text{cosec}^2 x - 1)\,dx$ $I_n = -\text{cosec}^{n-2}x\cot x - (n-2)I_n + (n-2)I_{n-2}$	dM1	1.1b
	$(n-1)I_n = -\text{cosec}^{n-2}x\cot x + (n-2)I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1}I_{n-2} - \frac{\text{cosec}^{n-2}x\cot x}{n-1}$ *	A1*	2.1
		(4)	
(b)	$I_6 = \frac{4}{5}I_4 - \frac{\text{cosec}^4 x\cot x}{5}$ or $[I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5}[I_4]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\text{cosec}^4 x\cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	1.1b
	$= \frac{4}{5}\left(\frac{2}{3}I_2 - \frac{\text{cosec}^2 x\cot x}{3}\right) - \frac{\text{cosec}^4 x\cot x}{5}$ or with limits etc	M1	1.1b
	$[I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{8}{15}[-\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{4\text{cosec}^2 x\cot x}{15}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\text{cosec}^4 x\cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	2.1
	$= \frac{8}{15}\left(\frac{\sqrt{3}}{3}\right) + \frac{16\sqrt{3}}{135} + \frac{16\sqrt{3}}{135} = \frac{56}{135}\sqrt{3}$ *	A1*	2.2a
		(4)	
			(8 marks)

Notes
(a) **For Alt 1 and any other similar approaches marking follows the same pattern**   M1: Splits the integrand into the product as shown and begins the process of integration by parts   For Alt 2 this requires applying the expression for $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$, splitting the integral and setting up the process for integration by parts on the composite term.   A1: Correct expression (for Alt 2 it is for a correct application of parts on their second term)   dM1: Depends on previous M. Applies $\cot^2 x = \pm 1 \pm \operatorname{cosec}^2 x$ and introduces I_n and I_{n-2} (For Alt 2 this is for complete substitution for I_n and I_{n-2})   A1*: Completes the proof by making I_n the subject with no errors seen (but condone minor notational slips). For Alt 1 a clear statement of replacing n by $n-2$ oe should be made.   (b)   M1: Begins process of application of reduction to find I_6 in terms of I_4 (need not evaluate terms) or deduces the value of I_2 (Alt 1)   M1: Uses the reduction formula correctly to find I_4 in terms of I_2 (need not be evaluated yet).   M1: A fully correct method using the reduction formula correctly to reach a value for I_6. Substitutions must be shown for the non-zero terms but accept decimals/trig functions for the M.   A1*: Reaches the printed answer with no errors, relevant working shown and trig terms evaluated

(a) **ALT 1**	$I_n = \int \operatorname{cosec}^{n+1} x \sin x \, dx$ (Allow with $n \pm 1$ in power for M's) $u = \operatorname{cosec}^{n+1} x, \frac{dv}{dx} = \sin x, \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$	M1	2.1
	$I_n = -\operatorname{cosec}^{n+1} x \cos x - (n+1) \int \operatorname{cosec}^n x (-\operatorname{cosec} x \cot x)(-\cos x) dx$	A1	1.1b
	$I_n = -\operatorname{cosec}^n x \cot x - (n+1) \int \operatorname{cosec}^n x \cot^2 x \, dx$		
	$I_n = -\operatorname{cosec}^n x \cot x - (n+1) \int \operatorname{cosec}^n x (\operatorname{cosec}^2 x - 1) dx$ $I_n = -\operatorname{cosec}^n x \cot x - (n+1) I_{n+2} + (n+1) I_n$	dM1	1.1b
	$(n+1) I_{n+2} = -\operatorname{cosec}^n x \cot x + n I_n$		
	replacing n by $n-2$ gives $I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$ *	A1*	2.1
		(4)	
(a) **ALT 2**	$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec}^{n-2} x (1 + \cot^2 x) dx$ $= \int \operatorname{cosec}^{n-2} x \, dx + \left(\int \operatorname{cosec}^{n-2} x \cot x \right) (\cot x) dx$ $u = \cot x, \frac{dv}{dx} = \operatorname{cosec}^{n-2} x \cot x, \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$	M1	2.1
	$I_n = I_{n-2} + \cot x \left(-\frac{\operatorname{cosec}^{n-2} x}{n-2} \right) - \int \left(-\frac{\operatorname{cosec}^{n-2} x}{n-2} \right) (-\operatorname{cosec}^2 x) dx$	A1	1.1b
	$(n-2) I_n = (n-2) I_{n-2} - \operatorname{cosec}^{n-2} x \cot x - \int \operatorname{cosec}^n x \, dx$		
	$(n-2) I_n = (n-2) I_{n-2} - \operatorname{cosec}^{n-2} x \cot x - I_n$	dM1	1.1b
	$(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$ *	A1*	2.1
		(4)	

(b) **ALT**	$I_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x \, dx = \left[-\cot x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{3}$	M1	2.2a
	$I_4 = \frac{2}{3} I_2 - \left[\frac{\operatorname{cosec}^2 x \cot x}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{2}{9}\sqrt{3} + \frac{4}{27}\sqrt{3}$	M1	1.1b
	$I_6 = \frac{4}{5} I_4 - \left[\frac{\operatorname{cosec}^4 x \cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5}\left(\frac{4}{27}\sqrt{3} + \frac{2}{9}\sqrt{3}\right) + \frac{16}{135}\sqrt{3}$	M1	2.1
	$= \frac{56}{135}\sqrt{3}$ *	A1*	1.1b
		(4)	

Q4.

Question	Scheme	Marks	AOs
	Complete overall strategy evidenced – requires finding the area of the two sides, and the area of the curved surface via attempt at the arc length first.	M1	3.1a
	Area of each side is $\int \frac{1}{2} r^2 \, d\theta = 450 \int_0^1 (1-\theta^2)^2 \, d\theta$	B1	1.1b
	$= 450 \int_0^1 1 - 2\theta^2 + \theta^4 \, d\theta = 450\left[\theta - \frac{2}{3}\theta^3 + \frac{1}{5}\theta^5\right]_0^1$	M1	1.1b
	$= 450\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 240 \text{ (cm}^2\text{)}$	A1	3.4
	$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 900\left(1 - 2\theta^2 + \theta^4\right) + \left(30 \times -2\theta\right)^2$	M1	1.1b
	$= 900\left(1+\theta^2\right)^2$	A1	2.2a
	Length of curve is $\int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = 30\int_0^1 1 + \theta^2 \, d\theta = 30\left[\theta + \frac{1}{3}\theta^3\right]_0^1$	M1	2.1
	$= 30\left(1 + \frac{1}{3} - (0)\right) = 40 \text{ (cm)}$	A1	3.4
	Surface area required is $2 \times "240" + \underline{235 \times "40"} = \ldots$	M1	1.1b
	$= 9880 \text{ cm}^2$	A1	3.2a
		(10)	
			(10 marks)

Notes:
M1: Shows a complete strategy for finding the required surface area – must include both sides, and attempt at area of curved surface using arc length using the correct formula.
B1: Uses polar area formula for at least one of the two sides. May use $2\times\int\frac{1}{2}r^2\,\mathrm{d}\theta$, but should be clear they are finding area of both sides. (Limits not needed for this mark.)
M1: Expands r^2 and integrates, powers to raise by 1.
A1: Applies limits and finds the area of one (or both) sides. 240 cm^2 for one sides, or 480 cm^2 for both. Look to see if they double when combining to see if they have one or two sides.
M1: Attempts $r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$ with correct differentiation. May be errors in squaring.
A1: Correct factorised expression, which may be implied by later work when they need to square root.
M1: Applies the arc length formula to their expression, must be a valid attempt to square root. (Limits not needed.)
A1: Applies the limits to the integral to obtain **correct** arc length.
M1: Uses area of curved surface is arc length $\times$ width of bump, with correct units used (not 2.35 and 40 unless they recover before adding) and adds the areas of the sides. Allow even if the attempt at the arc length came from incorrect application of the formula.
A1: cao 9880 cm^2

Q5.

Question	Scheme	Marks	AOs
(a)	$I_n = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx$	M1	2.1
	$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - (-)\int_0^{\frac{\pi}{2}} \cos^2 x (n-1)\sin^{n-2} x \, dx$	A1	1.1b
	Obtains $= 0 - (-)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)(n-1)\sin^{n-2} x \, dx$	M1	1.1b
	So $I_n = (n-1)I_{n-2} - (n-1)I_n$ and hence $nI_n = (n-1)I_{n-2}$ *	A1*	2.1
		(4)	
(b)	uses $I_n = \frac{(n-1)}{n} I_{n-2}$ to give $I_{10} = \frac{9}{10} I_8$ or $I_2 = \frac{1}{2} I_0$	M1	3.1b
	So $I_{10} = \frac{9\times7\times5\times3\times1}{10\times8\times6\times4\times2} I_0$	M1	2.1
	$I_0 = \frac{\pi}{2}$	B1	1.1b
	Required area is $2(I_2 - I_{10}) =$ or $8\times\frac{1}{4}(I_2 - I_{10}) =$	M1	3.1b
	$= 2\left(\frac{\pi}{4} - \frac{63\pi}{512}\right) = \frac{65\pi}{256} \text{ m}^2$	A1	1.1b
		(5)	
			(9 marks)

Notes:

(a)

M1: Splits the integrand into the product shown and begins process of integration by parts (there may be sign errors)

A1: Correct work

M1: Uses limits on the first term and expresses $\cos^2$ term in terms of $\sin^2$

A1*: Completes the proof collecting I_n terms correctly with all stages shown

(b)

M1: Attempts to find I_{10} and/or I_2

M1: Finds I_{10} in terms of I_0

B1: Finds I_0 correctly

M1: States the expression needed to find the required area

A1: Completes the calculation to give this exact answer

Q6.

Question Number	Scheme	Notes	Marks
(a) Way 1	$y=\ln\left(\frac{e^x+1}{e^x-1}\right)=\ln(e^x+1)-\ln(e^x-1)\Rightarrow\frac{dy}{dx}=\frac{e^x}{e^x+1}-\frac{e^x}{e^x-1}$ M1: Uses correct log rule and attempts derivative using chain rule A1: Correct Derivative		M1A1
	$=\frac{e^{2x}-e^x-e^{2x}-e^x}{e^{2x}-1}=\frac{-2e^x}{e^{2x}-1}$ *	dM1: Attempt single fraction and uses $(e^x-1)(e^x+1)=e^{2x}-1$. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1*
			(4)
	(a) Way 2		
	$\frac{dy}{dx}=\frac{e^x-1}{e^x+1}\left(\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}\right)$ Or $\frac{dy}{dx}=\frac{e^x-1}{e^x+1}\left(e^x(e^x-1)^{-1}-e^x(e^x+1)(e^x-1)^{-2}\right)$	M1: Uses chain and quotient or product rules A1: Correct derivative	M1A1
	$=\frac{1}{e^x+1}\left(-\frac{2e^x}{e^x-1}\right)=-\frac{2e^x}{e^{2x}-1}$ *	dM1: Cancels e^x-1 and uses $(e^x-1)(e^x+1)=e^{2x}-1$. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1*
	(a) Way 3		
	$y=\ln\left(\frac{e^x+1}{e^x-1}\right)\Rightarrow e^y=\frac{e^x+1}{e^x-1}\Rightarrow e^y\frac{dy}{dx}=\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}$ M1: Removes logs correctly and differentiates implicitly using chain and quotient rules A1: Correct differentiation		M1A1
	$\frac{dy}{dx}=-\frac{2e^x}{(e^x-1)^2}\times\frac{e^x-1}{e^x+1}=-\frac{2e^x}{e^{2x}-1}$ *	dM1: Divides by e^y in terms of x. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1

	(a) Way 4		
	$y=\ln\left(\frac{e^x+1}{e^x-1}\right)=\ln\left(\coth\frac{1}{2}x\right)\Rightarrow\frac{dy}{dx}=\frac{1}{\coth\frac{1}{2}x}\times-\frac{1}{2}\text{cosech}^2\frac{1}{2}x$ M1: Writes as $\ln\left(\coth\frac{1}{2}x\right)$ and differentiates using the chain rule A1: Correct differentiation		M1A1
	$=\left(\frac{e^x-1}{e^x+1}\right)\times\frac{-2e^x}{(e^x-1)^2}=-\frac{2e^x}{e^{2x}-1}$	dM1: Substitutes the correct exponential forms. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1

	(a) Way 5		
	$\text{artanh}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\Rightarrow y=2\text{artanh}\left(e^{-x}\right)$ $\frac{dy}{dx}=\frac{2}{1-\left(e^{-x}\right)^2}\times -e^{-x}$ M1: Writes y correctly in terms of artanh and attempts to differentiate using the chain rule A1: Correct differentiation		M1A1
	$\frac{dy}{dx}=\frac{-2e^{-x}}{1-e^{-2x}}=\frac{-2e^{x}}{e^{2x}-1}$ *	dM1: Multiplies numerator and denominator by e^{2x}. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1

	(a) Way 6		
	$y=\ln\left(1+\frac{2}{e^x-1}\right)\Rightarrow\frac{dy}{dx}=\frac{1}{1+2\left(e^x-1\right)^{-1}}\times -2e^x\left(e^x-1\right)^{-2}$ M1: Writes $\frac{e^x+1}{e^x-1}$ as $1+\frac{2}{e^x-1}$ and differentiates using the chain rule A1: Correct differentiation		M1A1
	$=\frac{-2e^x}{\left(e^x-1\right)^2+2\left(e^x-1\right)}=\frac{-2e^x}{e^{2x}-1}$	dM1: Multiplies denominator by $\left(e^x-1\right)^2$. **Dependent on the first method mark.** A1: Completes correctly with no errors	dM1A1

(b)	$L=\int\sqrt{1+\left(\pm\frac{2e^x}{e^{2x}-1}\right)^2}\,dx$	Uses the correct arc length formula with ± the result from part (a). **Note that we condone the omission of the minus sign on the fraction)**	M1
	$=\int\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}}\,dx$	Attempt single fraction. **Dependent on the first method mark.**	dM1
	Note that, for the first 2 marks, the candidate may just work on the integrand e.g. $\sqrt{1+\left(\pm\frac{2e^x}{e^{2x}-1}\right)^2}=\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}}$ **Would score the first 2 marks.**		
	$L=\int\sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\,dx=\int\frac{(e^{2x}+1)}{(e^{2x}-1)}\,dx$	Correct integral with square root removed. No limits required.	A1
	$=\int\coth x\,dx,\ \int\frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx,\ \int 1+\frac{2e^{-2x}}{1-e^{-2x}}\,dx,\ \frac{1}{2}\int\frac{2}{u}-\frac{1}{u+1}\,du\ (u=e^{2x}-1)$ $\frac{1}{2}\int\frac{2}{u-1}-\frac{1}{u}\,du\ (u=e^{2x}),\ \int\frac{1}{u+1}+\frac{1}{u-1}-\frac{1}{u}\,du\ (u=e^x),$ $\frac{1}{2}\int\frac{2}{u-2}-\frac{1}{u-1}\,du\ (u=e^{2x}+1)$		
	$=[\ln\sinh x],\left[\ln(e^x-e^{-x})\right],\left[x+\ln(1-e^{-2x})\right],\left[\ln u-\ln\sqrt{(1+u)}\right],$ $\left[\ln(u-1)-\ln\sqrt{u}\right],\left[\ln\frac{(u^2-1)}{u}\right],\left[\ln(u-2)-\ln\sqrt{u-1}\right]$ Correct integration		A1
	$=\ln\sinh(\ln 3)-\ln\sinh(\ln 2)$ $\left(=\ln\frac{4}{3}-\ln\frac{3}{4}\right)$	Correct use of limits e.g. ln3 and ln2 for x and e.g. 3 and 8 if $u=e^{2x}-1$. They must be the correct limits for their method if they use substitution. **Dependent on both previous method marks.**	ddM1
	$=\ln\frac{16}{9}$	cao	A1
			(6)
			Total 10

Q7.

Question Number	Scheme	Notes	Marks
	$I_n = \int_0^{\ln 2} \cosh^n x \, dx$		
(a)	$I_n = \int \cosh^{n-1} x \cosh x \, dx$		
	$I_n = \int \cosh^{n-1} x \cosh x \, dx = \sinh x \cosh^{n-1} x - \int (n-1)\cosh^{n-2} x \sinh^2 x \, dx$ M1: Integration by parts in the correct direction. If the formula is quoted it must be correct otherwise look for an expression of the form $\pm \sinh x \cosh^{n-1} x \pm k \int \cosh^{n-2} x \sinh^2 x \, dx$ A1: Correct expression		M1A1
	$= \sinh x \cosh^{n-1} x - \int (n-1)\cosh^{n-2} x (\cosh^2 x - 1) \, dx$	Replaces $\sinh^2 x$ with $\pm\cosh^2 x \pm 1$ on the "integration part" to obtain an expression in $\cosh x$ only. **Dependent on the first method mark.**	dM1
	$= \sinh x \cosh^{n-1} x - (n-1)\int \cosh^n x \, dx + (n-1)\int \cosh^{n-2} x \, dx$		
	$= \sinh x \cosh^{n-1} x - (n-1) I_n + (n-1) I_{n-2}$	Introduces I_n and I_{n-2}. **Dependent on both previous method marks.**	ddM1
	$\left[\sinh x \cosh^{n-1} x\right]_0^{\ln 2} = \sinh(\ln 2)\cosh^{n-1}(\ln 2)(-0)$ $\left(= \left(\frac{3}{4}\right)\left(\frac{5}{4}\right)^{n-1} \right)$	Use of given limits on their $\sinh x \cosh^{n-1} x$. Does not need to be evaluated but note that $\cosh(\ln 2) = \frac{5}{4}, \sinh(\ln 2) = \frac{3}{4}$	M1
	$I_n = \frac{3 \times 5^{n-1}}{n \times 4^n} + \frac{(n-1)}{n} I_{n-2}$ *	cao	A1*
			(6)

	(a) Way 2		
	$I_n = \int \cosh^{n-2} x \cosh^2 x \, dx = \int \cosh^{n-2} x \, dx + \int \cosh^{n-2} x \sinh^2 x \, dx$ Writes $\cosh^n x$ as $\cosh^{n-2} x \cosh^2 x$ and uses $\sinh^2 x = \pm\cosh^2 x \pm 1$		M1
	$\int \cosh^{n-2} x \sinh^2 x \, dx = \left[\frac{\sinh x \cosh^{n-1} x}{n-1}\right] - \frac{1}{n-1}\int \cosh^n x \, dx$ M1: Integration by parts in the correct direction. If the formula is quoted it must be correct otherwise look for an expression of the form $p \sinh x \cosh^{n-1} x \pm q \int \cosh^n x \, dx$ A1: Correct expression		dM1A1
	$(n-1) I_n = (n-1) I_{n-2} + \left[\sinh x \cosh^{n-1} x\right] - I_n$	Introduces I_n and I_{n-2}. **Dependent on both previous method marks.**	ddM1
	$\left[\sinh x \cosh^{n-1} x\right]_0^{\ln 2} = \sinh(\ln 2)\cosh^{n-1}(\ln 2)(-0)$ $\left(= \left(\frac{3}{4}\right)\left(\frac{5}{4}\right)^{n-1} \right)$	Use of given limits on their $\sinh x \cosh^{n-1} x$. Does not need to be evaluated but note that $\cosh(\ln 2) = \frac{5}{4}, \sinh(\ln 2) = \frac{3}{4}$	M1
	$I_n = \frac{3 \times 5^{n-1}}{n \times 4^n} + \frac{(n-1)}{n} I_{n-2}$ *	cao	A1*
	You can condone the occasional missing x, dx and limits along the way and "invisible" brackets may be recovered. **Do not allow e.g. an obvious sign error that gets "corrected" later – withhold the final A1 in such cases.**		

(b)	$I_4 = \frac{3\times5^3}{4\times4^4} + \frac{3}{4}I_2$ or $\frac{3\times a^3}{4\times b^4} + \frac{3}{4}I_2$	Correct first application of **their or the given** reduction formula	M1
	$= \frac{3\times5^3}{4\times4^4} + \frac{3}{4}\left(\frac{3\times5}{2\times4^2} + \frac{1}{2}I_0\right)$ or $\frac{3\times a^3}{4\times b^4} + \frac{3}{4}\left(\frac{3\times a}{2\times b^2} + \frac{1}{2}I_0\right)$ Correct second application of **their or the given** reduction formula **that is consistent with the formula used in the first application** to obtain I_4 in terms of I_0		M1
	$I_0 = \ln 2$		B1
	$I_4 = \frac{735}{1024} + \frac{3}{8}\ln 2$	Cao (Allow equivalent exact forms e.g. may be factorised but fractions must be collected)	A1
	Note that candidates may work from the "other end" e.g. $I_0 = \ln 2$ B1 $I_2 = \frac{3\times5}{2\times4^2} + \frac{1}{2}I_0$ M1 *I_2 in terms of I_0* $I_4 = \frac{3\times5^3}{4\times4^4} + \frac{3}{4}\left(\frac{3\times5}{2\times4^2} + \frac{1}{2}I_0\right)$ M1 *I_4 in terms of I_0* $I_4 = \frac{735}{1024} + \frac{3}{8}\ln 2$ A1 Cao (Allow equivalent exact forms e.g. may be factorised but fractions must be collected)		
			(4)
(b) Way 2	$I_4 = \frac{3\times5^3}{4\times4^4} + \frac{3}{4}I_2$	Correct application of their reduction formula	M1
	$I_2 = \int_0^{\ln 2} \cosh^2 x\,dx = \int_0^{\ln 2}\left(\frac{1}{2} + \frac{1}{2}\cosh 2x\right)dx$		
	$\int\left(\frac{1}{2} + \frac{1}{2}\cosh 2x\right)dx = \frac{x}{2} + \frac{1}{4}\sinh 2x$	Correct integration	B1
	$I_2 = \left[\frac{x}{2} + \frac{1}{4}\sinh 2x\right]_0^{\ln 2} = \frac{1}{2}\ln 2 + \frac{15}{32}$	Correct use of limits on an expression of the form $\alpha x + \beta\sinh 2x$	M1
	$I_4 = \frac{3\times5^3}{4\times4^4} + \frac{3}{4}\left(\frac{1}{2}\ln 2 + \frac{15}{32}\right)$		
	$I_4 = \frac{735}{1024} + \frac{3}{8}\ln 2$	Cao (Allow equivalent exact forms e.g. may be factorised but fractions must be collected)	A1

(b) Way 3	$I_4 = \int_0^{\ln 2} \cosh^4 x \, dx = \int_0^{\ln 2} \left(\frac{1}{2} + \frac{1}{2}\cosh 2x\right)^2 dx$		
	$\int_0^{\ln 2} \left(\frac{1}{4} + \frac{1}{2}\cosh 2x + \frac{1}{4}\cosh^2 2x\right) dx$	$\cosh^4 x = \frac{1}{4} + \frac{1}{2}\cosh 2x + \frac{1}{4}\cosh^2 2x$	B1
	$\frac{1}{4}\int_0^{\ln 2} \left(1 + 2\cosh 2x + \frac{1}{2}(1 + \cosh 4x)\right) dx$	$\cosh^2 2x = \pm\frac{1}{2} \pm \frac{1}{2}\cosh 4x$ and attempt to integrate	M1
	$\frac{1}{4}\left[\frac{3x}{2} + \sinh 2x + \frac{1}{8}\sinh 4x\right]_0^{\ln 2}$	Correct use of correct limits	M1
	$I_4 = \frac{735}{1024} + \frac{3}{8}\ln 2$	Cao (Allow equivalent exact forms e.g. may be factorised but fractions must be collected)	A1

(b) Way 4	$I_4 = \int_0^{\ln 2} \cosh^4 x \, dx = \int_0^{\ln 2} \left(\frac{e^x + e^{-x}}{2}\right)^4 dx$		
	$= \int_0^{\ln 2} \left(\frac{e^x + e^{-x}}{2}\right)^4 dx = \left(\frac{1}{16}\right)\int_0^{\ln 2} \left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right) dx$ Correct expansion		B1
	$= \left(\frac{1}{16}\right)\left[\frac{e^{4x}}{4} + 2e^{2x} + 6x - 2e^{-2x} - \frac{e^{-4x}}{4}\right]_0^{\ln 2}$	Attempts to integrate their expansion	M1
	$\left(\frac{1}{16}\right)\left[\left(4 + 8 + 6\ln 2 - \frac{1}{2} - \frac{1}{64}\right) - (0)\right]_0^{\ln 2}$	Correct use of correct limits	M1
	$I_4 = \frac{735}{1024} + \frac{3}{8}\ln 2$	Cao (Allow equivalent exact forms e.g. may be factorised but fractions must be collected)	A1
			Total 10

Q8.

Question	Scheme	Marks	AOs
(a)	$\frac{dy}{dx} = -\sinh 2x$	B1	2.1
	so $S = \int \sqrt{1+\sinh^2 2x}\,dx$	M1	2.1
	$\therefore S = \int \cosh 2x\,dx$	A1	1.1b
	$= [\frac{1}{2}\sinh 2x]_{-\ln a}^{\ln a}$ or $[\sinh 2x]_0^{\ln a}$	M1	2.1
	$= \sinh 2\ln a = \frac{1}{2}[e^{2\ln a} - e^{-2\ln a}] = \frac{1}{2}\left(a^2 - \frac{1}{a^2}\right)$ (so k = ½)	A1	1.1b
		(5)	
(b)	$\frac{1}{2}\left(a^2 - \frac{1}{a^2}\right) = 2$ so $a^4 - 4a^2 - 1 = 0$	M1	1.1b
	$a^2 = 2 + \sqrt{5}$ (and $a = 2.06$ (approx.))	M1	1.1b
	When $x = \ln a$, $y = 0$ so $A = \frac{1}{2}\cosh(2\ln a)$	M1	3.4
	Height = $A - 0.5$ = awrt 0.62m	A1	1.1b
		(4)	
(c)	The width of the base = $2\ln a$ = 1.4m	B1	3.4
		(1)	
(d)	A parabola of the form $y = 0.62 - 1.19x^2$, or other symmetric curve with its equation e.g. $0.62\cos(2.2x)$	M1A1	3.3 3.3
		(2)	
			(12 marks)

Notes:

(a)
B1: Starts explanation by finding the correct derivative
M1: Uses their derivative in the formula for arc length
A1: Uses suitable identity to simplify the integrand and to obtain the expression in scheme
M1: Integrates and uses appropriate limits to find the required arc length
A1: Uses the definition of sinh to complete the proof and identifies the value for k

(b)
M1: Uses the formula obtained from the model and the length of the arch to create a quartic equation
M1: Continues to use this model to obtain a quadratic and to obtain values for a
M1: Attempts to find a value for A in order to find h
A1: Finds a value for the height correct to 2sf (or accept exact answer)

Notes: (continued)
(c) **B1:** Finds width to 2 sf i.e. 1.4m
(d) **M1:** Chooses or describes an even function with maximum point on the y axis **A1:** Gives suitable equation passing through (0, 0.62) and (0.7, 0) and (- 0.7, 0)

Q9.

Question	Scheme	Marks	AOs
(a)	$I_n = \int t^{n-1} \times t\sqrt{4+5t^2}\,\mathrm{d}t = t^{n-1} \times K\left(4+5t^2\right)^{\frac{3}{2}} - \int (n-1)t^{n-2} \times K\left(4+5t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	M1	3.1a
	$I_n = t^{n-1} \times \frac{2}{3\times 10}\left(4+5t^2\right)^{\frac{3}{2}} - \int (n-1)t^{n-2} \times \frac{2}{3\times 10}\left(4+5t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	A1	1.1b
	$= t^{n-1} \times \frac{1}{15}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{(n-1)}{15}\int t^{n-2}\left(4+5t^2\right)^{\frac{1}{2}} \times \left(4+5t^2\right)\mathrm{d}t$ $= \frac{t^{n-1}}{15}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{4(n-1)}{15}\int t^{n-2}\left(4+5t^2\right)^{\frac{1}{2}}\,\mathrm{d}t - \frac{5(n-1)}{15}\int t^{n}\left(4+5t\right)^{\frac{1}{2}}\,\mathrm{d}t$	M1	3.1a
	$\Rightarrow 15I_n = t^{n-1}\left(4+5t^2\right)^{\frac{3}{2}} - 4(n-1)I_{n-2} - 5(n-1)I_n \Rightarrow I_n = \ldots$	M1	1.1b
	$I_n = \frac{t^{n-1}}{5(n+2)}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)}I_{n-2}$ *	A1*	2.1
		(5)	
(b)	Surface area $= 2\pi\int_0^1 y\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$	B1	1.1a
	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{5}{\sqrt{5}}t^4$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2t^3$	B1	1.1b
	$\int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int \frac{1}{2}t^4\sqrt{\left(\frac{5}{\sqrt{5}}t^4\right)^2 + \left(2t^3\right)^2}\,\mathrm{d}t$	M1	1.1b
	$= \int \frac{1}{2}t^4\sqrt{5t^8+4t^6}\,\mathrm{d}t = \frac{1}{2}\int t^7\sqrt{4+5t^2}\,\mathrm{d}t$	M1	2.1
	Hence surface area $= \pi\int_0^1 t^7\sqrt{4+5t^2}\,\mathrm{d}t$ *	A1*	1.1b
		(5)	

(c)	$[I_1]_0^1 = \left[\frac{1}{15}(4+5t^2)^{\frac{3}{2}}\right]_0^1 = \frac{27}{15} - \frac{8}{15} = \frac{19}{15}$ (=1.266...)	B1	2.2a
	$\int_0^1 t^7\sqrt{4+5t^2}\,dt = \left[\frac{t^6}{5\times9}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times6}{5\times9}[I_5]_0^1$	M1	1.1b
	$=\frac{3}{5}-\frac{8}{15}\left(\left[\frac{t^4}{5\times7}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times4}{5\times7}[I_3]_0^1\right)$ $=\frac{3}{5}-\frac{8}{15}\left(\frac{27}{35}-\frac{16}{35}\left(\left[\frac{t^2}{5\times5}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times2}{5\times5}[I_1]_0^1\right)\right)$	M1	3.1a
	Total surface area is $(\pi)\left[\frac{3}{5}-\frac{8}{15}\left(\frac{27}{35}-\frac{16}{35}\left(\frac{27}{25}-\frac{8}{25}\times\frac{19}{15}\right)\right)\right]=...$	M1	2.1
	=awrt 1.11 (3sf) $\left(=\frac{69509\,\pi}{196875}\right)$	A1	1.1b
		(5)	
	For the three method marks if the process is worked the other way: $[I_3]_0^1 = \left[\frac{t^2}{5\times5}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times2}{5\times5}[I_1]_0^1\left(=\frac{27}{25}-\frac{8\times19}{25\times35}=\frac{253}{375}=0.6746...\right)$	M1	1.1b
	$[I_5]_0^1 = \left[\frac{t^4}{5\times7}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times4}{5\times7}[I_3]_0^1\left(=\frac{27}{35}-\frac{16}{35}\times\frac{253}{375}=\frac{6077}{13125}=0.4630...\right)$ $[I_7]_0^1 = \left[\frac{t^6}{5\times9}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \frac{4\times6}{5\times9}[I_5]_0^1 = ...$	M1	3.1a
	$=\frac{27}{45}-\frac{24}{45}\left(\frac{6077}{13125}\right)=...$	M1	2.1
	=awrt 1.11	A1	1.1b
			(15 marks)

Notes:
(a) **M1:** Splits the integrand correctly and applies integration by parts in the correct direction to achieve a form as shown in the scheme. **A1:** Correct result of applying parts, need not be simplified. **M1:** Splits the integrand to identify I_n and I_{n-2} (or allow if I_{n-1} appears due to error for this mark) in the equation. **M1:** Rearranges to make I_n the subject from an equation in I_n and I_{n-2} **A1*:** Correct completion to the given result.
(b) **B1:** Correct parametric formula for surface area given. Must include the 2π and limits, but these may be added at a later stage and must the 2π must be seen before cancelling occurs. **B1:** Correct derivatives of x and y with respect to t seen or implied. **M1:** Applies their derivatives and y to $\int y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$. May have included the limits and 2π here, but they are not needed for this mark. **M1:** Squares the derivatives and takes a common factor t^3 from the square root to reach appropriate form for the integral. Limits and 2π not needed for this mark. **A1*:** Reaches correct answer with no errors seen, limits included (but do not need to be justified) and the dt must be present and the 2π must have been seen and correctly processed. (c) **B1:** Correct value for I_1 between the limits - need not be simplified and may be seen later in the working. **M1:** Applies the reduction formula from (a) in attempt to solve the integral. This may be from I_7 to I_5 or from I_1 to I_3 depending on the direction they are going. Allow for any application relevant to the integral (e.g between two odd values for n).

M1: Applies the reduction formula two more times to link I_1 and I_7. May have evaluated at each stage or find expression before substituting limits but look for the complete process to link the two intervals. **dM1:** Applies the limits to their integral in a complete process to reach an answer. Allow if substitution happens throughout the process of reduction or at the end but it must be a complete process to find reach a value, though allow if the π is not included. **A1:** Must have scored all three method marks. Correct answer, awrt 1.11.