Hyperbolic Functions

Questions

Q1.

Given that y = arsinh(tanh x), show that

$$\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1 + \tanh^2 x}}$$

(5)

(Total for question = 5 marks)

Q2.

$$f(x) = \frac{1}{\sqrt{4x^2 + 9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)dx = A \sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence find, in exact form in terms of natural logarithms, the mean value of f(x) over the interval [0, 3].

(2)

(Total for question = 6 marks)

Q3.

The curve C has equation

$$y = 31 \sinh x - 2 \sinh 2x \qquad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C.

(Total for question = 7 marks)

Q4.

(a) Using the definition for cosh x in terms of exponentials, show that

$$\cosh 2x \equiv 2\cosh^2 x - 1$$

(3)

(b) Find the exact values of x for which

$$29\cosh x - 3\cosh 2x = 38$$

giving your answers in terms of natural logarithms.

(6)

(Total for question = 9 marks)

Q5.

(a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant k.

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)

(Total for question = 10 marks)

Q6.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Determine the values of x for which

$$64\cosh^4 x - 64\cosh^2 x - 9 = 0$$

Give your answers in the form $q \ln 2$ where q is rational and in simplest form.

(Total for question = 4 marks)

Q7.

Solutions based entirely on graphical or numerical methods are not acceptable.

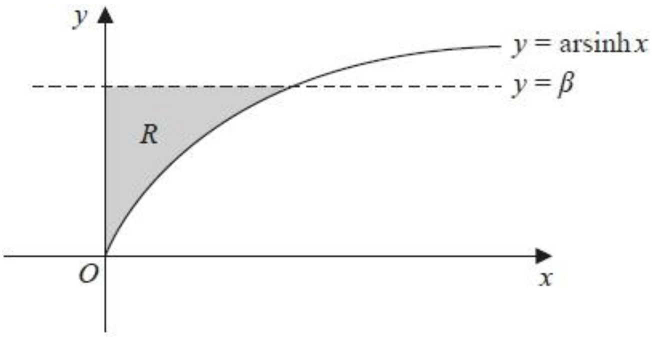

Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \operatorname{arsinh} x \qquad x \geq 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \operatorname{arsinh} x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

(Total for question = 9 marks)

Q8.

(a) Use a hyperbolic substitution and calculus to show that

$$\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right] + k$$

where k is an arbitrary constant.

(6)

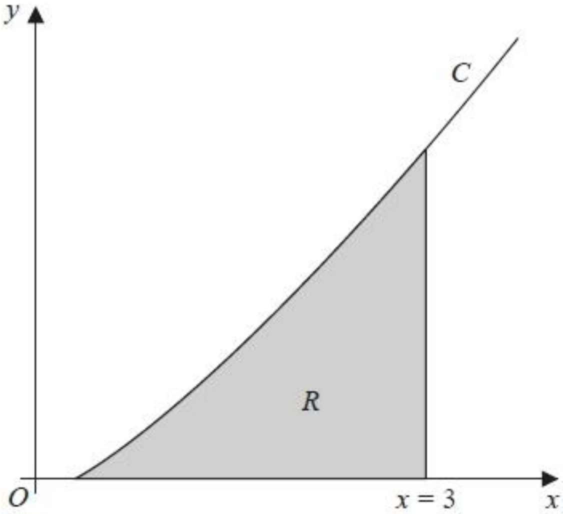

Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{4}{15}x \operatorname{arcosh} x \qquad x \geqslant 1$$

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Using algebraic integration and the result from part (a), show that the area of R is given by

$$\frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$$

(5)

(Total for question = 11 marks)

Q9.

(i) (a) Explain why $\int_0^{\infty} \cosh x \, \mathrm{d}x$ is an improper integral.

(1)

(b) Show that $\int_0^{\infty} \cosh x \, \mathrm{d}x$ is divergent.

(3)

(ii) $$4 \sinh x = p \cosh x \qquad \text{where } p \text{ is a real constant}$$

Given that this equation has real solutions, determine the range of possible values for p

(2)

(Total for question = 6 marks)

Mark Scheme – Hyperbolic Functions

Q1.

Question Number	Scheme	Notes	Marks
	$y = \text{arsinh}(\tanh x)$		
Way 1	$\sinh y = \tanh x$		B1
	$\cosh y \frac{dy}{dx} = \text{sech}^2 x$ or $\cosh y = \text{sech}^2 x \frac{dx}{dy}$	M1: $\pm\cosh y$ or $\pm\text{sech}^2 x$ A1: All correct	M1A1
	$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\cosh y}$		
	$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\sqrt{1+\sinh^2 y}} = f(x)$	Uses a **correct identity** to express $\frac{dy}{dx}$ in terms of x only	M1
	$= \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}}$ *	cso. There must be no errors such as incorrect or missing or inconsistent variables and no missing h's.	A1*
			Total 5
Way 2	$t = \tanh x \Rightarrow y = \text{arsinh} t$	Replaces $\tanh x$ by e.g. t	B1
	$\frac{dt}{dx} = \text{sech}^2 x, \frac{dy}{dt} = \frac{1}{\sqrt{1+t^2}}$	M1: $\frac{dt}{dx} = \pm\text{sech}^2 x, \frac{dy}{dt} = \pm\frac{1}{\sqrt{1+t^2}}$ A1: Correct $\frac{dt}{dx}$ and $\frac{dy}{dt}$ and correctly labelled	M1A1
	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\text{sech}^2 x}{\sqrt{1+t^2}} = f(x)$	Uses **correct form of the chain rule** for their variables to express $\frac{dy}{dx}$ in terms of x only	M1
	$= \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}}$ *	Cso. There must be no errors such as incorrect or missing or inconsistent variables and no missing h's.	A1*
			Total 5
Way 3	$u = \tanh x \Rightarrow \frac{du}{dx} = \text{sech}^2 x$	Correct derivative	B1
	$\int \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}} dx = \int \frac{\text{sech}^2 x}{\sqrt{1+u^2}} \frac{1}{\text{sech}^2 x} du$	M1: Complete substitution including the "dx" A1: Fully correct substitution	M1A1
	$= \int \frac{1}{\sqrt{1+u^2}} du = \text{arsinh} u (+c)$	Reaches $\text{arsinh} u$	M1
	$y = \text{arsinh}(\tanh x)(+c)$	Reaches $y = \text{arsinh}(\tanh x)$ with or without + c and no errors such as incorrect or missing or inconsistent variables or missing h's.	A1*
			Total 5
	Special Case:		
	$y = \text{arsinh}(\tanh x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+\tanh^2 x}}(\times)\text{sech}^2 x$ $= \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}}$ **Note that the sech²x needs to appear separate from the fraction as above <u>and not just the printed answer written down</u>.** **<u>To score more than 2 marks using a chain rule method, a third variable must be introduced</u>**		M1A1

Q2.

Question	Scheme	Marks	AOs
(a) Way 1	$x=\frac{3}{2}\sinh u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\sinh^2 u+9}}\times\frac{3}{2}\cosh u\,du$	M1	3.1a
	$=\int \frac{1}{2}\,du$	A1	1.1b
	$=\int \frac{1}{2}\,du=\frac{1}{2}u=\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)+c$	A1	1.1b
		(4)	
(a) Way 2	$x=\frac{3}{2}\tan u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\tan^2 u+9}}\times\frac{3}{2}\sec^2 u\,du$	M1	3.1a
	$=\int \frac{1}{2}\sec u\,du$	A1	1.1b
	$=\frac{1}{2}\ln\left(\sec u+\tan u\right)=\frac{1}{2}\ln\left(\frac{2x}{3}+\sqrt{1+\left(\frac{2x}{3}\right)^2}\right)$ $u=\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)+c$	A1	1.1b
(a) Way 3	$x=\frac{1}{2}u$ or $x=ku$ where $k>0$ $k\neq 1$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{1}{4}\right)u^2+9}}\times\frac{1}{2}du$	M1	3.1a
	$=\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}du\quad\left(\text{or } \frac{1}{2}\int \frac{1}{\sqrt{u^2+\frac{9}{4k^2}}}du \text{ for } x=ku\right)$	A1	1.1b
	$=\frac{1}{2}\sinh^{-1}\frac{u}{3}=\frac{1}{2}\sinh^{-1}\frac{2x}{3}+c$	A1	1.1b

Question	Scheme	Marks	AOs
(b)	Mean value = $\frac{1}{3(-0)}\left[\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)\right]_0^3=\frac{1}{3}\times\frac{1}{2}\sinh^{-1}\left(\frac{2\times 3}{3}\right)(-0)$	M1	2.1
	$=\frac{1}{6}\ln\left(2+\sqrt{5}\right)$ **(Brackets are required)**	A1ft	1.1b
		(2)	
		(6 marks)	

Notes
(a) B1: Selects an appropriate substitution leading to an integrable form M1: Demonstrates a fully correct method for the substitution that includes substituting into the function and dealing with the "dx". The substitution being substituted does not need to be "correct" for this mark but the substitution must be an attempt at $\int \frac{1}{\sqrt{4[\mathrm{f}(u)]^2+9}} \times \mathrm{f}'(u)\,\mathrm{d}u$ with the $\mathrm{f}'(u)$ correct for their substitution. E.g. if $x = \frac{1}{2}u$ is used, must see $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ not $2\mathrm{d}u$. A1: Correct simplified integral in terms of u from correct work and from a correct substitution A1: Correct answer including "$+ c$". Allow arcsinh or arsinh for $\sinh^{-1}$ from correct work and from a correct substitution (b) M1: Correctly applies the method for the mean value for their integration which must be of the form specified in part (a) and substitutes the limits 0 and 3 but condone omission of 0 A1: Correct exact answer (follow through their A and B). **Brackets are required if appropriate.**

Q3.

Question	Scheme	Marks	AOs
	$\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$	B1	1.1b
	$\frac{dy}{dx} = 31\cosh x - 4\left(2\cosh^2 x - 1\right)$	M1	3.1a
	$8\cosh^2 x - 31\cosh x - 4 = 0$	A1	1.1b
	$(8\cosh x + 1)(\cosh x - 4) = 0 \Rightarrow \cosh = \ldots$	M1	1.1b
	$\cosh x = 4, \left(-\frac{1}{8}\right)$	A1	1.1b
	$\cosh x = \alpha \Rightarrow x = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $-\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha - \sqrt{\alpha^2 - 1}\right)$ or $\frac{e^x + e^{-x}}{2} = 4\text{Þ}\ e^{2x} - 8e^x + 7 = 0\text{Þ}\ e^x = \ldots\text{Þ}\ x = \ln(\ldots)$	M1	1.2
	$\pm\ln\left(4 + \sqrt{15}\right)$ or $\ln\left(4 \pm \sqrt{15}\right)$	A1	2.2a
		(7)	

	Alternative $\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$ or $31\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right)$	B1	1.1b
	Using $\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$ and $\sinh x = \left(\frac{e^x + e^{-x}}{2}\right)$ as required $\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 0$ leading to $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ o.e.	M1 A1	3.1a 1.1b
	Solves $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ $\text{Þ}\ e^x = \ldots$	M1	1.1b
	$e^x = 4 \pm \sqrt{15}$ or awrt 7.87, 0.13	A1	1.1b
	$x = \ln(b)$ where b is a real exact value	M1	1.2
	$\ln\left(4 \pm \sqrt{15}\right)$	A1	2.2a
		(7)	
			(7 marks)

Notes
B1: Correct differentiation M1: Identifies a correct approach by using a correct identity to make progress to obtain a quadratic in $\cosh x$ A1: Correct 3 term quadratic obtained M1: Solves their 3TQ A1: Correct values (may only see 4 here) M1: Correct process to reach at least one value for x from their $\cosh x$ A1: Deduces the correct 2 values with no incorrect values or work involving $\cosh x = -\frac{1}{8}$
Alternative B1: Correct differentiation M1: Using the exponential form for $\cosh x$, and $\sinh x$ if required, and forms a quartic equation for e^x with all terms simplified and all on one side A1: Correct quartic equation for e^x M1: Solves their quartic equation in e^x A1: Correct values to two decimal places or exact values M1: $x = \ln(b)$ where b is a real exact value A1: Deduces the correct 2 values only

Q4.

Question Number	Scheme	Notes	Marks
	$\cosh 2x \equiv 2\cosh^2 x - 1$		
	Note that exponentials must be used in (a)		
(a) Way 1	$\text{rhs} = 2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$	Substitutes the **correct** exponential form into the rhs	M1
	$= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1$	**Squares** correctly to obtain an expression in e^{2x} and e^{-2x}. **Dependent on the previous mark.**	dM1
	$= \frac{e^{2x} + e^{-2x}}{2} + 1 - 1$		
	$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x = \text{lhs}^*$	Complete proof with no errors	A1*
			(3)
	(a) Way 2		
	$\text{lhs} = \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$	Substitutes the correct exponential form	M1
	$= 2\left(\frac{(e^x + e^{-x})^2 - 2}{4}\right)$	**Completes the square** correctly to obtain an expression in e^x and e^{-x} **Dependent on the previous mark.**	dM1
	$2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = 2\cosh^2 x - 1 = \text{rhs}^*$	Complete proof with no errors	A1*

(b) Way 1	$29\cosh x-3\left(2\cosh^2 x-1\right)=38$	Substitutes the result from part (a)	M1
	$6\cosh^2 x-29\cosh x+35=0\Rightarrow\cosh x=...$	Forms a 3-term quadratic and attempt to solve for $\cosh x$. You can apply the General Principles for solving a 3TQ if necessary.	M1
	$\cosh x=\frac{7}{3}$ or $\cosh x=\frac{5}{2}$	Both correct (or equivalent values)	A1
	$\cosh x=\alpha\Rightarrow x=\ln\left(\alpha+\sqrt{\alpha^2-1}\right)$ or $\cosh x=\alpha\Rightarrow x=\ln\left(\alpha-\sqrt{\alpha^2-1}\right)$ or $\frac{e^x+e^{-x}}{2}=\alpha\Rightarrow x=...$	Uses the correct ln form for arcosh to find at least one value for x for $\alpha>1$ or uses the correct exponential form for cosh and solves the resulting 3TQ in e^x to find at least one value for x for $\alpha>1$	M1
	$x=\ln\left(\frac{7}{3}\pm\sqrt{\frac{40}{9}}\right)$ and $x=\ln\left(\frac{5}{2}\pm\sqrt{\frac{21}{4}}\right)$ Or equivalent exact forms e.g. $x=\ln\frac{7\pm2\sqrt{10}}{3}$ and $x=\ln\frac{5\pm\sqrt{21}}{2}$ $x=\pm\ln\left(\frac{7+2\sqrt{10}}{3}\right)$ and $x=\pm\ln\left(\frac{5+\sqrt{21}}{2}\right)$ $x=\ln\left(7\pm2\sqrt{10}\right)-\ln3$ and $x=\ln\left(5\pm\sqrt{21}\right)-\ln2$ **A1: Any 2 of these 4 solutions. Penalise lack of brackets once where necessary, the first time it occurs and penalise lack of simplification once, the first time it occurs** e.g. $\ln\frac{5}{2}\pm\frac{\sqrt{21}}{2}$, $\ln\left(\frac{5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-1}\right)$ **A1: All 4 correct**		A1A1
	Note that the decimal answers are, ±1.49…, ±1.56…,		
			(6)
			Total 9

	(b) Way 2		
	$29\left(\frac{e^x+e^{-x}}{2}\right)-3\left(\frac{e^{2x}+e^{-2x}}{2}\right)=38$ or $6\left(\frac{e^x+e^{-x}}{2}\right)^2-29\left(\frac{e^x+e^{-x}}{2}\right)+35=0$	Substitutes the correct exponential forms	M1
	$3e^{4x}-29e^{3x}+76e^{2x}-29e^x+3=0$	M1: Multiplies by e^{2x} or e^{-2x} to obtain a quartic in e^x or e^{-x} A1: Correct quartic in any form (not necessarily all on one side)	M1A1
	$\left(3e^{2x}-14e^x+3\right)\left(e^{2x}-5e^x+1\right)=0\Rightarrow x=...$	Solves their quartic to find at least one value for x	M1
	$x=\ln\left(\frac{7}{3}\pm\sqrt{\frac{40}{9}}\right)$ and $x=\ln\left(\frac{5}{2}\pm\sqrt{\frac{21}{4}}\right)$ Or equivalent exact forms e.g. $x=\ln\frac{7\pm2\sqrt{10}}{3}$ and $x=\ln\frac{5\pm\sqrt{21}}{2}$ $x=\pm\ln\left(\frac{7+2\sqrt{10}}{3}\right)$ and $x=\pm\ln\left(\frac{5+\sqrt{21}}{2}\right)$ $x=\ln\left(7\pm2\sqrt{10}\right)-\ln3$ and $x=\ln\left(5\pm\sqrt{21}\right)-\ln2$ e.g. $\ln\frac{5}{2}\pm\frac{\sqrt{21}}{2}$, $\ln\left(\frac{5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-1}\right)$ **A1: All 4 correct**		A1A1

Q5.

Question	Scheme	Marks	AOs
(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y = x \Rightarrow x=\dfrac{\sinh y}{\cosh y}=\dfrac{e^y-e^{-y}}{e^y+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ or $\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ Allow this to score M1A1		
	$x(e^{2y}+1)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\dfrac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$*	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^y: $e^y=\dfrac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\dfrac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\dfrac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\dfrac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\dfrac{1}{2}\ln\dfrac{(x+1)}{(1-x)}$* Score **M1** for an attempt at the quadratic formula to make e^y the subject (condone $\pm\sqrt{\dots}$) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k=1$ or $-1<x<1$	B1	1.1b
		(5)	

(a) Way 2	$\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow x=\tanh\left(\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\right)=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$	M1 A1	2.1 1.1b
	$x=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}=\dfrac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}=x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)=\dfrac{1}{2}\ln(2-3x)\Rightarrow\dfrac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^2-9x+1=0$	A1	1.1b
	$6x^2-9x+1=0\Rightarrow x=\dots$	M1	1.1b
	$x=\dfrac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	
	Alternative for first 2 marks of (b)		
	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow 2x=\dfrac{e^{2\ln\sqrt{2-3x}}-1}{e^{2\ln\sqrt{2-3x}}+1}$	M1	3.1a
	$\Rightarrow\dfrac{2-3x-1}{2-3x+1}=2x$	M1	2.1

Notes
(a) **<u>If you come across any attempts to use calculus to prove the result – send to review</u>** M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials. The exponential form can be any of $\frac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2}$, $\frac{e^y - e^{-y}}{e^y + e^{-y}}$, $\frac{e^{2y}-1}{e^{2y}+1}$ Allow any variables to be used **but the final answer must be in terms of** x. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh. A1: Correct expression for "x" in terms of exponentials M1: Full method to make $e^{2"y"}$ the subject of the formula. This must be correct algebra so allow sign errors only. A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors. Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$, $\frac{1}{2}\ln\frac{x+1}{1-x}$, $\frac{1}{2}\ln\left\|\frac{x+1}{1-x}\right\|$. Need to see $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion but allow if the proof concludes that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier. B1: Correct value for k or writes $-1 < x < 1$ **Way 2** M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials A1: Correct expression M1: Eliminates exponentials and logs and simplifies A1: Correct result (i.e. $x = x$) with conclusion B1: Correct value for k or writes $-1 < x < 1$ (b) M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides M1: Makes the link with part (a) by replacing artanh($2x$) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed **correctly**. A1: Obtains the correct 3TQ M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1) A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \frac{9 \pm \sqrt{57}}{12}$ scores A0 unless the positive root is rejected **Alternative for first 2 marks of (b)** M1: Adopts a correct strategy by expressing tanh in terms of exponentials M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly

Q6.

Question	Scheme	Marks	AOs
	Solves the quadratic equation for $cosh^2 x$ e.g. $(8\,cosh^2 x - 9)(8\,cosh^2 + 1) = 0 \Rightarrow cosh^2 x = \ldots$	M1	3.1a
	$cosh^2 x = \frac{9}{8} \left\{-\frac{1}{8}\right\}$	A1	1.1b
	$cosh\, x = \frac{3}{4}\sqrt{2} \Rightarrow x = ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\left(\frac{3}{4}\sqrt{2}\right)^2 - 1}\right]$ **Alternatively** $cosh\, x = \frac{3}{4}\sqrt{2} \Rightarrow \frac{1}{2}(e^x + e^{-x}) \Rightarrow e^{2x} - \frac{3}{2}\sqrt{2}e^x + 1 = 0$ $\Rightarrow e^x = \sqrt{2}$ or $\frac{\sqrt{2}}{2} \Rightarrow x = \ldots$	M1	1.1b
	$x = \pm\frac{1}{2} ln\, 2$	A1	2.2a
		(4)	
		(4 marks)	

Notes:

M1: Solves the quadratic equation for $cosh^2 x$by any valid means. If by calculator accept for reaching the positive value for $cosh^2 x$ (negative may be omitted or incorrect) but do not allow for going directly to a value for $cosh\, x$ Alternatively score a correct process leading to a value for sinh $2x$ or its square (Alt 1) or use of correct exponential form for cosh x to form and expand to an equation in e^{4x} and e^{2x} (Alt 2)

A1: Correct value for $cosh^2 x$ (ignore negative or incorrect extra roots.). In Alt 1 score for a correct value for $\sinh^2 2x$ or sinh $2x$. In Alt 2 score for a correct simplified equation in e^{4x}.

M1: For a correct method to achieve at least one value for x (from $\cosh^2 x$). In the main scheme or Alt 1, takes positive square root (if appropriate) and uses the correct formula for arcosh x or arsinh x to find a value for x. (No need to see negative square root rejected.) In Alt 2 it is for solving the quadratic in e^{4x} and proceeding to find a value for x.

Alternatively uses the exponential definition for cosh x, forms and solves a quadratic for e^x leading to a value for x

A1: Deduces (both) the correct values for x and no others. Must be in the form specified.

SC Allow M0A0M1A1 for cases where a calculator was used to get the value for cosh x with no evidence if a correct method for find both values is shown.

Alt 1	$64\cosh^2 x\left(\cosh^2 x-1\right)-9=0 \Rightarrow 64\cosh^2 x\sinh^2 x-9=0$ $\Rightarrow 16\sinh^2 2x=9 \Rightarrow \sinh^2 2x=\dfrac{9}{16}$ Or $(8\sinh x\cosh x-3)(8\sinh x\cosh x+3)=0 \Rightarrow \sinh 2x=\pm\frac{3}{4}$	M1 A1	3.1a 1.1b
	$\sinh 2x=\pm\frac{3}{4} \Rightarrow x=\frac{1}{2}\ln\left[\pm\frac{3}{4}+\sqrt{\frac{9}{16}+1}\right]$ (or use exponentials, or proceed via cosh4x)	M1	1.1b
	$x=\pm\dfrac{1}{2}\ln 2$	A1	2.2a
		(4)	
Alt 2	$64\left(\dfrac{e^x+e^{-x}}{2}\right)^4-64\left(\dfrac{e^x+e^{-x}}{2}\right)^2-9=0 \Rightarrow$ $4(e^{4x}+4e^{2x}+6+4e^{-2x}+e^{-4x})-16(e^{2x}+2+e^{-2x})-9=0$	M1	3.1a
	$4e^{4x}-17+4e^{-4x}=0$	A1	1.1b
	$(4e^{4x}-1)(1-4e^{-4x})=0 \Rightarrow e^{4x}=\ldots \Rightarrow x=\ldots$	M1	1.1b
	$x=\pm\dfrac{1}{2}\ln 2$	A1	2.2a
		(4)	

Q7.

Question	Scheme		Marks	AOs
(a)	Using $\text{arsinh}\,\alpha = \frac{1}{2}\ln 3$ $\alpha = \frac{e^{\frac{1}{2}\ln 3} - e^{-\frac{1}{2}\ln 3}}{2}$	$\ln\left(\alpha + \sqrt{\alpha^2+1}\right) = \frac{1}{2}\ln 3$	B1	1.2
	$\alpha = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{2} \Rightarrow \alpha = \ldots$	$\alpha + \sqrt{\alpha^2+1} = \sqrt{3}$ $\sqrt{\alpha^2+1} = \sqrt{3} - \alpha$ $\alpha^2 + 1 = 3 - 2\sqrt{3}\alpha + \alpha^2 \Rightarrow \alpha = \ldots$	M1	1.1b
	$\alpha = \frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$		A1	2.2a
			(3)	
(b)	Volume = $\pi\int_0^{\frac{1}{2}\ln 3} \sinh^2 y\, dy$		B1	2.5
	$\{\pi\}\int\left(\frac{e^y - e^{-y}}{2}\right)^2 dy = \{\pi\}\int\left(\frac{e^{2y} - 2 + e^{-2y}}{4}\right) dy$ or $\{\pi\}\int \frac{1}{2}\cosh 2y - \frac{1}{2}\, dy$		M1	3.1a
	$\frac{1}{4}\left(\frac{1}{2}e^{2y} - 2y - \frac{1}{2}e^{-2y}\right)$ or $\frac{1}{4}\sinh 2y - \frac{1}{2}y$		dM1 A1	1.1b 1.1b
	Use limits $y = 0$ and $y = \frac{1}{2}\ln 3$ and subtracts the correct way round		M1	1.1b
	$\frac{\pi}{4}\left(\frac{4}{3} - \ln 3\right)$ or exact equivalent		A1	1.1b
			(6)	
			(9 marks)	

Notes:

(a)

B1: Recalls the definition for $\sinh\left(\frac{1}{2}\ln 3\right)$ or forms an equation for arcsinh x

M1: Uses logarithms to find a value for α or forms and solves a correct equation without log

A1: Deduces the correct exact value for α

Note using the result

$$\ln\left(\frac{1}{\sqrt{3}} + \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}\right) = \ln\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{4}{3}}\right) = \ln\sqrt{3} = \frac{1}{2}\ln 3 \text{ therefore } \text{arsinh}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\ln 3$$

B1 for substituting in α into arcsinhx, M1 for rearranging to show $\frac{1}{2}\ln 3$, A1 for conclusion
(b) **B1**: Correct expression for the volume $\pi\int_0^{\frac{1}{2}\ln 3}\sinh^2 y\,dy$ requires integration signs, dy and correct limits. **M1**: Uses the exponential formula for sinh y or the identity $\cosh 2y = \pm 1 \pm 2\sinh^2 y$ to write in a form which can be integrated at least one term **dM1**: Dependent of previous method mark, integrates. **A1**: Correct integration. **M1**: Correct use of the limits $y = 0$ and $y = \frac{1}{2}\ln 3$ **A1**: Correct exact volume.

Q8.

Question	Scheme	Marks	AOs
(a)	$\int \frac{x^2}{\sqrt{x^2-1}}\,dx \to \int f(u)\,du$ Uses the substitution $x = \cosh u$ fully to achieve an integral in terms of u only, including replacing the dx	M1	3.1a
	$\int \frac{\cosh^2 u}{\sqrt{\cosh^2 u - 1}} \sinh u\,(du)$	A1	1.1b
	Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 2\cosh^2 u - 1$ to achieve an integral of the form $A\int(\cosh 2u \pm 1)\,du \qquad A > 0$	M1	3.1a
	Integrates to achieve $A\left(\pm\frac{1}{2}\sinh 2u \pm u\right)(+c) \qquad A > 0$	M1	1.1b
	Uses the identity $\sinh 2u = 2\sinh u \cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ $\to \sinh 2u = 2x\sqrt{x^2-1}$	M1	2.1
	$\frac{1}{2}\left[x\sqrt{x^2-1} + \text{ar}\cosh x\right] + k$ * cso	A1*	1.1b
		(6)	

(b)	Uses integration by parts the correct way around to achieve $\int \frac{4}{15} x \operatorname{arcosh} x \, dx = Px^2 \operatorname{arcosh} x - Q\int \frac{x^2}{\sqrt{x^2-1}} dx$	M1	2.1
	$= \frac{4}{15}\left(\frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2-1}} dx\right)$	A1	1.1b
	$= \frac{4}{15}\left(\frac{1}{2}x^2 \operatorname{arcosh} x - \frac{1}{2}\left(\frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right]\right)\right)$	B1ft	2.2a
	Uses the limits $x = 1$ and $x = 3$ the correct way around and subtracts $= \frac{4}{15}\left(\frac{1}{2}(3)^2 \operatorname{arcosh} 3 - \frac{1}{2}\left(\frac{1}{2}\left[3\sqrt{(3)^2-1} + \operatorname{arcosh} 3\right]\right)\right) - \frac{4}{15}(0)$	dM1	1.1b
	$= \frac{4}{15}\left(\frac{9}{2}\ln\left(3+\sqrt{8}\right) - \frac{3\sqrt{8}}{4} - \frac{1}{4}\ln\left(3+\sqrt{8}\right)\right)$ $= \frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$*	A1*	1.1b
		(5)	
			(11 marks)

Notes:

(a)

M1: Uses the substitution $x = \cosh u$ fully to achieve an integral in terms of u only. Must have replaced the dx but allow if the du is missing.

A1: Correct integral in terms of u. (Allow if the du is missing.)

M1: Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 2\cosh^2 u - 1$ to achieve an integrand of the required form

M1: Integrates to achieve the correct form, may be sign errors.

M1: Uses the identities $\sinh 2u = 2\sinh u \cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ to attempt to find $\sinh 2u$ in terms of x. If using exponentials there must be a full and complete method to attempt the correct form.

A1*: Achieves the printed answer with no errors seen, cso

NB attempts at integration by parts are not likely to make progress – to do so would need to split the integrand as $x\frac{x}{\sqrt{x^2-1}}$. If you see any attempts that you feel merit credit, use review.

(b)

M1: Uses integration by parts the correct way around to achieve the required form.

A1: Correct integration by parts

B1ft: Deduces the integral by using the result from part (a). Follow through on their 'uv'

dM1: Dependent on previous method mark. Uses the limits $x = 1$ and $x = 3$ the correct way around and subtracts

A1*cso: Achieves the printed answer with at least one intermediate step showing the evaluation of the arcosh 3, and no errors seen.

Q9.

Question	Scheme	Marks	AOs
(i) (a)	E. g. • Because the interval being integrated over is unbounded. • $\cosh x$ is undefined at the limit of ∞ • the upper limit is infinite	B1	1.2
		(1)	
(i) (b)	$\int_0^{\infty} \cosh x \,dx = \lim_{t\to\infty} \int_0^t \cosh x \,dx$ or $\lim_{t\to\infty} \int_0^t \frac{1}{2}(e^x + e^{-x})dx$	B1	2.5
	$\int_0^t \cosh x \,dx = [\sinh x]_0^t = \sinh t\ (-0)$ or $\frac{1}{2}\int_0^t e^x + e^{-x}\,dx = \frac{1}{2}[e^x - e^{-x}]_0^t = \frac{1}{2}[e^t - e^{-t}]\left(-\frac{1}{2}[e^0 - e^0]\right)$	M1	1.1b
	When $t \to \infty$ $e^t \to \infty$and $e^{-t} \to 0$ therefore the integral is divergent	A1	2.4
		(3)	
(ii)	$4\sinh x = p\cosh x \Rightarrow \tanh x = \frac{p}{4}$ or $4\tanh x = p$ **Alternative** $\frac{4}{2}(e^x - e^{-x}) = \frac{p}{2}(e^x + e^{-x}) \Rightarrow 4e^x - 4e^{-x} = pe^x + pe^{-x}$ $e^{2x}(4-p) = p + 4 \Rightarrow e^{2x} = \frac{p+4}{4-p}$	M1	3.1a
	$\left\{-1 < \frac{p}{4} < 1 \Rightarrow\right\} -4 < p < 4$	A1	2.2a
		(2)	
			(6 marks)

(i)(a) **B1**: For a suitable explanation. Technically this should refer to the interval being unbounded, but this is unlikely to be seen. Accept "Because the upper limit is infinity", but **not** "because it is infinity" without reference to what "it" is. Do not accept "the upper limit tends to infinity" or "the integral is unbounded".
(i)(b) **B1**: Writes the integral in terms of a limit as $t \to \infty$ (or other variable) with limits 0 and "t", or implies the integral is a limit by subsequent working by correct language. **M1**: Integrates $\cosh x$ correctly either as $\sinh x$ or in terms of exponentials and applies correctly the limits of 0 and "t". The bottom limit zero may be implied. No need for the $\lim_{t\to\infty}$ for this mark but substitution of ∞ is M0. **A1**: cso States that (as $t \to \infty$) $\sinh t \to \infty$ or $e^t \to \infty$and $e^{-t} \to 0$ therefore divergent (or not convergent), or equivalent working. Accept $\sinh t$ is undefined as $t \to \infty$
(ii) **M1**: Divides through by $\cosh x$ to find an expression involving $\tanh x$ Alternative: uses the correct exponential definitions and finds an expression for e^{2x} or solves a quadratic in e^{2x} **A1**: Deduces the correct inequality for p. Note $\|p\| < 4$ is a correct inequality for p.