# Roots of Polynomials

## Questions

**Q1.**

The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where $p$, $q$ and $r$ are integers to be found.

**(5)**

**(Total for question = 5 marks)**

**Q2.**

The cubic equation

$$3x^3 + x^2 - 4x + 1 = 0$$

has roots $\alpha, \beta,$ and $\gamma$.

Without solving the cubic equation,

(a) determine the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

**(3)**

(b) find a cubic equation that has roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma},$ giving your answer in the form $x^3 + ax^2 + bx + c = 0$, where $a$, $b$ and $c$ are integers to be determined.

**(3)**

**(Total for question = 6 marks)**

**Q3.**

The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where $p$, $q$ and $r$ are integers to be found.

**(5)**

**(Total for question = 5 marks)**

**Q4.**

The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where $p$, $q$, $r$ and $s$ are integers to be found.

**(Total for question = 5 marks)**

**Q5.**

The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where $a$, $b$, $c$ and $d$ are integers to be determined.

**(Total for question = 5 marks)**

**Q6.**

The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are $\alpha$, $\beta$, $\gamma$ and $\delta$

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

**(3)**

(ii) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} + \frac{2}{\delta}$

**(3)**

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

**(3)**

**(Total for question = 9 marks)**

# Mark Scheme – Roots of Polynomials

Q1.

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $w = 2z+1 \Rightarrow z = \frac{w-1}{2}$ | B1 | 3.1a |
| | $\left(\frac{w-1}{2}\right)^3 - 3\left(\frac{w-1}{2}\right)^2 + \left(\frac{w-1}{2}\right) + 5 = 0$ | M1 | 3.1a |
| | $\frac{1}{8}\left(w^3 - 3w^2 + 3w - 1\right) - \frac{3}{4}\left(w^2 - 2w + 1\right) + \frac{w-1}{2} + 5 = 0$ | | |
| | $w^3 - 9w^2 + 19w + 29 = 0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | (5) | |
| ALT 1 | $\alpha + \beta + \gamma = 3, \alpha\beta + \beta\gamma + \alpha\gamma = 1, \alpha\beta\gamma = -5$ | B1 | 3.1a |
| | New sum = $2(\alpha+\beta+\gamma)+3 = 9$ | M1 | 3.1a |
| | New pair sum = $4(\alpha\beta+\beta\gamma+\gamma\alpha)+4(\alpha+\beta+\gamma)+3 = 19$ | | |
| | New product = $8\alpha\beta\gamma+4(\alpha\beta+\beta\gamma+\gamma\alpha)+2(\alpha+\beta+\gamma)+1 = -29$ | | |
| | $w^3 - 9w^2 + 19w + 29 = 0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | (5) | |
| | | | **(5 marks)** |

**Notes**

B1: Selects the method of making a connection between $z$ and $w$ by writing $z = \frac{w-1}{2}$

M1: Applies the process of substituting their $z = \frac{w-1}{2}$ into $z^3 - 3z^2 + z + 5 = 0$

(Allow $z = 2w + 1$)

M1: Manipulates their equation into the form $w^3 + pw^2 + qw + r (= 0)$ having substituted their $z$ in terms of $w$. Note that the "= 0" can be missing for this mark.

A1: At least two of $p$, $q$, $r$ correct. Note that the "= 0" can be missing for this mark.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of $w$ but the final answer must be in terms of $w$.

**ALT1**

B1: Selects the method of giving three correct equations containing α, β and γ

M1: Applies the process of finding the new sum, new pair sum, new product

M1: Applies $w^3 - (\text{new sum})w^2 + (\text{new pair sum})w - (\text{new product})(= 0)$

or identifies $p$ as –(new sum) $q$ as (new pair sum) and $r$ as –(new product)

A1: At least two of $p$, $q$, $r$ correct.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of $w$ but the final answer must be in terms of $w$.

**Q2.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $\alpha\beta\gamma = -\frac{1}{3}$ and $\alpha\beta+\alpha\gamma+\beta\gamma = -\frac{4}{3}$ | B1 | 3.1a |
| | $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} = \frac{-\frac{4}{3}}{-\frac{1}{3}}$ | M1 | 1.1b |
| | $=4$ | A1 | 1.1b |
| | | (3) | |
| (b) | $\left\{\alpha+\beta+\gamma = -\frac{1}{3}\right\}$<br>New product $= \frac{1}{\alpha}\times\frac{1}{\beta}\times\frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = \frac{1}{-\frac{1}{3}} = \ldots(-3)$<br>New pair sum $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{-\frac{1}{3}}{-\frac{1}{3}} = \ldots(1)$ | M1 | 3.1a |
| | $x^3 - (\text{part (a)})x^2 + (\text{new pair sum})x - (\text{new product})(=0)$ | M1 | 1.1b |
| | $x^3-4x^2+x+3=0$ | A1 | 1.1b |
| | | (3) | |
| | **Alternative**<br>e.g. $z=\frac{1}{x} \Rightarrow \frac{3}{x^3}+\frac{1}{x^2}-\frac{4}{x}+1=0$ | M1 | 3.1a |
| | $x^3-4x^2+x+3=0$ | M1<br>A1 | 1.1b<br>1.1b |
| | | (3) | |
| | | | **(6 marks)** |

**Notes:**

**(a)**

**B1:** Correct values for the product and pair sum of the roots

**M1:** A complete method to find the sum of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. Must substitute in their values of the product and pair sum

**A1:** correct value 4

Note: If candidate does not divide by 3 so that $\alpha\beta\gamma=-1$ and $\alpha\beta+\alpha\gamma+\beta\gamma=-4$ the maximum they can score is B0 M1 A0

**(b)**

**M1:** A correct method to find the value of the new pair sum and the value of the new product

**M1:** Applies $x^3 - (\text{part (a)})x^2 + (\text{their new pair sum})x - (\text{their new product})(=0)$

**A1:** Fully correct equation, in any variable, including = 0

**(b) Alternative**

**M1:** Realises the connection between the roots and substitutes into the cubic equation

**M1:** Manipulates their equation into the form $x^3+ax^2+bx+c=0$

**A1:** Fully correct equation in any variable, including = 0

**Q3.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $\{w = x - 1 \Rightarrow\}\ x = w + 1$ | B1 | 3.1a |
| | $(w+1)^3 + 3(w+1)^2 - 8(w+1) + 6 = 0$ | M1 | 3.1a |
| | $w^3 + 3w^2 + 3w + 1 + 3(w^2 + 2w + 1) - 8w - 8 + 6 = 0$ | | |
| | $w^3 + 6w^2 + w + 2 = 0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | (5) | |
| ALT 1 | $\alpha + \beta + \gamma = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -8,\ \alpha\beta\gamma = -6$ | B1 | 3.1a |
| | sum roots = $\alpha - 1 + \beta - 1 + \gamma - 1$<br>$= \alpha + \beta + \gamma - 3 = -3 - 3 = -6$<br>pair sum = $(\alpha-1)(\beta-1) + (\alpha-1)(\gamma-1) + (\beta-1)(\gamma-1)$<br>$= \alpha\beta + \alpha\gamma + \beta\gamma - 2(\alpha+\beta+\gamma) + 3$<br>$= -8 - 2(-3) + 3 = 1$<br>product = $(\alpha - 1)(\beta - 1)(\gamma - 1)$<br>$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha+\beta+\gamma) - 1$<br>$= -6 - (-8) - 3 - 1 = -2$ | M1 | 3.1a |
| | $w^3 + 6w^2 + w + 2 = 0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | (5) | |
| | | **(5 marks)** | |

| | | Question Notes |
|---|---|---|
| | B1 | Selects the method of making a connection between $x$ and $w$ by writing $x = w + 1$ |
| | M1 | Applies the process of substituting their $x = w + 1$ into $x^3 + 3x^2 - 8x + 6 = 0$ |
| | M1 | Depends on previous M mark. Manipulating their equation into the form $w^3 + pw^2 + qw + r = 0$ |
| | A1 | At least two of $p$, $q$, $r$ are correct. |
| | A1 | Correct final equation. |
| ALT 1 | B1 | Selects the method of giving three correct equations each containing $\alpha$, $\beta$ and $\gamma$. |
| | M1 | Applies the process of finding sum roots, pair sum and product. |
| | M1 | Depends on previous M mark. Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - \text{their } \alpha\beta\gamma = 0$ |
| | A1 | At least two of $p$, $q$, $r$ are correct. |
| | A1 | Correct final equation. |

**Q4.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $\{w = x+3 \Rightarrow\}\ x = w-3$ | B1 | 3.1a |
| | $2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (= 0)$ | M1 | 1.1b |
| | $2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (= 0)$ | | |
| | $2w^3 - 12w^2 + 15w + 21 = 0$<br>(So $p = 2$, $q = -12$, $r = 15$ and $s = 21$) | M1<br>A1<br>A1 | 3.1a<br>1.1b<br>1.1b |
| | | (5) | |

| | | | |
|---|---|---|---|
| ALT 1 | $\alpha + \beta + \gamma = -\frac{6}{2} = -3$, $\alpha\beta + \beta\gamma + \alpha\gamma = -\frac{3}{2}$, $\alpha\beta\gamma = -\frac{12}{2} = -6$ | B1 | 3.1a |
| | sum roots $= \alpha + 3 + \beta + 3 + \gamma + 3$<br>$= \alpha + \beta + \gamma + 9 = -3 + 9 = 6$<br>pair sum $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$<br>$= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha+\beta+\gamma) + 27$<br>$= -\frac{3}{2} + 6\times -3 + 27 = \frac{15}{2}$<br>product $= (\alpha+3)(\beta+3)(\gamma+3)$<br>$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha+\beta+\gamma) + 27$<br>$= -6 + 3\times -\frac{3}{2} + 9\times -3 + 27 = -\frac{21}{2}$ | M1 | 3.1a |
| | $w^3 - 6w^2 + \frac{15}{2}w - \left(-\frac{21}{2}\right)\ (= 0)$ | M1 | 1.1b |
| | $2w^3 - 12w^2 + 15w + 21 = 0$<br>(So $p = 2$, $q = -12$, $r = 15$ and $s = 21$) | A1<br>A1 | 1.1b<br>1.1b |
| | | (5) | |
| | | **(5 marks)** | |

| Notes | | |
|---|---|---|
| | B1 | Selects the method of making a connection between $x$ and $w$ by writing $x = w-3$ |
| | M1 | Applies the process of substituting their $x = aw \pm b$ into $2x^3 + 6x^2 - 3x + 12\ (= 0)$<br>So accept e.g. if $x = \frac{w}{3}$ is used. |
| | M1 | Depends on having attempted substituting either $x = w-3$ or $x = w+3$ into the equation. This mark is for manipulating their resulting equation into the form $pw^3 + qw^2 + rw + s (= 0)$ ($p \neq 0$). The "= 0" may be implied for this. |
| See note | A1 | At least three of $p$, $q$, $r$ and $s$ are correct in an equation with integer coefficients. (need not have "= 0") |
| | A1 | Correct final equation, including "=0". Accept integer multiples. |
| ALT 1 | B1 | Selects the method of giving three correct equations each containing $\alpha$, $\beta$ and $\gamma$. |
| | M1 | Applies the process of finding sum roots, pair sum and product. |
| | M1 | Applies $w^3 -$ (their sum roots)$w^2 +$ (their pair sum)$w -$ (their product) $(= 0)$<br>Must be correct identities, but if quoted allow slips in substitution, but the "=0" may be implied. |
| See note | A1 | At least three of $p$, $q$, $r$ and $s$ are correct in an equation with integer coefficients. (need not have "=0") |
| | A1 | Correct final equation, including "=0". Accept multiples with integer coefficients. |
| Note: may use another variable than $w$ for the first four marks, but the final equation must be in terms of $w$ | | |
| **Notes: Do not isw the final two A marks – if subsequent division by 2 occurs then mark the final answer.** | | |

**Q5.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $w = 3x - 2 \Rightarrow x = \frac{w+2}{3}$ | B1 | 3.1a |
| | $9\left(\frac{w+2}{3}\right)^3 - 5\left(\frac{w+2}{3}\right)^2 + 4\left(\frac{w+2}{3}\right) + 7 = 0$ | M1 | 3.1a |
| | $\frac{1}{3}\left(w^3 + 6w^2 + 12w + 8\right) - \frac{5}{9}\left(w^2 + 4w + 4\right) + \frac{4}{3}(w+2) + 7 = 0$ | | |
| | $3w^3 + 13w^2 + 28w + 91 = 0$ | dM1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | **(5)** | |
| | **Alternative:** | | |
| | $\alpha + \beta + \gamma = \frac{5}{9}, \alpha\beta + \beta\gamma + \alpha\gamma = \frac{4}{9}, \alpha\beta\gamma = -\frac{7}{9}$ | B1 | 3.1a |
| | New sum $= 3(\alpha + \beta + \gamma) - 6 = -\frac{13}{3}$ | M1 | 3.1a |
| | New pair sum $= 9(\alpha\beta + \beta\gamma + \gamma\alpha) - 12(\alpha + \beta + \gamma) + 12 = \frac{28}{3}$ | | |
| | New product $= 27\alpha\beta\gamma - 18(\alpha\beta + \beta\gamma + \gamma\alpha) + 12(\alpha + \beta + \gamma) - 8 = -\frac{91}{3}$ | | |
| | $w^3 - \left(-\frac{13}{3}\right)w^2 + \frac{28}{3}w - \left(-\frac{91}{3}\right) = 0$ | dM1 | 1.1b |
| | $3w^3 + 13w^2 + 28w + 91 = 0$ | A1<br>A1 | 1.1b<br>1.1b |
| | | **(5)** | |
| | | | **(5 marks)** |

| Notes |
|---|
| B1: Selects the method of making a connection between $x$ and $w$ by writing $x = \frac{w+2}{3}$<br>Condone the use of a different letter than $w$<br>M1: Applies the process of substituting $x = \frac{w+2}{3}$ into $9x^3 - 5x^2 + 4x + 7 = 0$<br>dM1: Depends on the previous M mark. Manipulates their equation into the form $aw^3 + bw^2 + cw + d\,(=0)$. Condone the use of a different letter then $w$ consistent with B1 mark.<br>A1: At least two of $a, b, c, d$ correct<br>A1: Fully correct equation, must be in terms of $w$<br>**Alternative:**<br>B1: Selects the method of giving three correct equations containing $\alpha$, $\beta$ and $\gamma$<br>M1: Applies the process of finding the new sum, new pair sum, new product<br>dM1: Depends on the previous M mark. Applies<br>$w^3 - (\text{new sum})w^2 + (\text{new pair sum})w - (\text{new product})(=0)$ condone the use of any letter here.<br>A1: At least two of $a, b, c, d$ correct<br>A1: Fully correct equation in term of $w$ |

**Q6.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (i) | $\sum \alpha_i = -\frac{5}{3}$ and $\sum \alpha_i\alpha_j = 0$<br>This mark can be awarded if seen in part (ii) or part (iii) | B1 | 3.1a |
| | So $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2\left(\sum \alpha_i\alpha_j\right) = \ldots$ | M1 | 1.1b |
| | $= \frac{25}{9} - 2\times 0 = \frac{25}{9}$ | A1 | 1.1b |
| | | (3) | |
| (ii) | $\sum \alpha_i\alpha_j\alpha_k = \frac{7}{3}$ and $\prod \alpha_i = 2$ or for $x = \frac{2}{w}$ used in equation<br>This mark can be awarded if seen in part (i) or part (iii) | B1 | 2.2a |
| | So $2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) = 2\times\frac{\sum \alpha_i\alpha_j\alpha_k}{\alpha\beta\gamma\delta} = 2\times\frac{'\frac{7}{3}'}{'\frac{6}{3}'}$ or for<br>$3\left(\frac{16}{w^4}\right)+5\left(\frac{8}{w^3}\right)-7\left(\frac{2}{w}\right)+6=0 \Rightarrow 6w^4-14w^3+\ldots=0$ leading to $\frac{14}{6}$ | M1 | 1.1b |
| | $\left(=2\times\frac{7/3}{2}\right)\left(=\frac{14}{6}\right)=\frac{7}{3}$ | A1 | 1.1b |
| | | (3) | |

| | | | |
|---|---|---|---|
| (iii) | $(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = \ldots$ expands all four brackets<br>Or equation with these roots is $3(3-x)^4+5(3-x)^3-7(3-x)+6=0$ | M1 | 3.1a |
| | $= 81-27\left(\sum\alpha_i\right)+9\left(\sum\alpha_i\alpha_j\right)-3\left(\sum\alpha_i\alpha_j\alpha_k\right)+\prod\alpha_i$<br>$= 81-27\left(-\frac{5}{3}\right)+9(0)-3\left(\frac{7}{3}\right)+2$<br>Or expands to fourth power and constant terms and attempts product of roots $3x^4+\ldots+3\times3^4+5\times3^3-7\times3+6 \to \prod\alpha_i = \frac{"363"}{3}$ | dM1 | 1.1b |
| | $=121$ | A1 | 1.1b |
| | | (3) | |
| | | (9 marks) | |

**Notes:**

(i)

**B1:** Correct sum and pair sum of roots seen or implied. Must realise the pair sum is zero.

**Note:** These values can be seen anywhere in the candidate's solution

**M1:** Uses correct expression for the sum of squares.

**A1:** $\frac{25}{9}$. Allow this mark from incorrect sign on sum of squares (but they will score B0 if the sign is incorrect).

(ii)

**B1:** Correct triple sum and product of roots seen or implied. May be stated in (i). Alternatively, this may be scored for sight of $x = \frac{2}{w}$ used as a transformation in the equation.

**Note:** These values can be seen anywhere in the candidate's solution

**M1:** Substitutes their values into $2 \times \frac{\sum \alpha_i \alpha_j \alpha_k}{\alpha\beta\gamma\delta} = ...$ In the alternative it is for rearranging the equation to a quartic in $w$ and uses to find the sum of the roots.

**A1:** $\frac{7}{3}$ Allow this mark from incorrect sign of both triple sum and product (but they will score B0 if the sign is incorrect).

**(iii)**

**M1:** A correct method to find the value used – may recognise structure as scheme, may expand the expression in stages, or may attempt to use a linear transformation $(3 - x)$ or e.g. $(3 - w)$ in original equation. Condone slips as long as the intention is clear.

**dM1:** Dependent on previous method mark. Uses at least 2 values of their sum of roots etc. in their expression. If using a linear shift this is for expanding to find the coefficient of $x^4$ and constant term and attempts product of roots by dividing the constant term by the coefficient of $x^4$.

**A1:** 121.