Combinations of Random Variables

Questions

Q1.

Sugar is packed into medium bags and large bags. The weights of the medium bags of sugar are normally distributed with mean 520 grams and standard deviation 10 grams. The weights of the large bags of sugar are normally distributed with mean 1510 grams and standard deviation 20 grams.

(a) Find the probability that a randomly chosen large bag of sugar weighs at least 15 grams more than the combined weight of 3 randomly chosen medium bags of sugar.

(6)

(b) Find the probability that a randomly chosen large bag of sugar weighs less than 3 times the weight of a randomly chosen medium bag of sugar.

(5)

A random sample of 5 medium bags of sugar is taken.

(c) Find the value of d so that the probability that all 5 bags of sugar each weigh more than 520 grams is equal to the probability that the mean weight of the 5 bags of sugar is more than d grams.

(5)

(Total for question = 16 marks)

Q2.

A manufacturer makes two versions of a toy. One version is made out of wood and the other is made out of plastic.

The weights, W kg, of the wooden toys are normally distributed with mean 2.5 kg and standard deviation 0.7 kg. The weights, X kg, of the plastic toys are normally distributed with mean 1.27 kg and standard deviation 0.4 kg. The random variables W and X are independent.

(a) Find the probability that the weight of a randomly chosen wooden toy is more than double the weight of a randomly chosen plastic toy.

(6)

The manufacturer packs n of these wooden toys and $2n$ of these plastic toys into the same container. The maximum weight the container can hold is 252 kg.

The probability of the contents of this container being overweight is 0.2119 to 4 decimal places.

(b) Calculate the value of n.

(8)

(Total for question = 14 marks)

Q3.

The weights of a particular type of apple, A grams, and a particular type of orange, R grams, each follow independent normal distributions.

$$A \sim \mathrm{N}(160, 12^2) \qquad R \sim \mathrm{N}(140, 10^2)$$

(a) Find the distribution of

(i) $A + R$
(ii) the total weight of 2 randomly selected apples.

(3)

A box contains 4 apples and 1 orange only. Jesse selects 2 pieces of fruit at random from the box.

(b) Find the probability that the total weight of the 2 pieces of fruit exceeds 310 grams.

(3)

From a large number of apples and oranges, Celeste selects m apples and 1 orange at random. The random variable W is given by

$$W = \left(\sum_{i=1}^{m} A_i\right) - n \times R$$

where n is a positive integer.

Given that the middle 95% of the distribution of W lies between 1100.08 and 1499.92 grams,

(c) find the value of m and the value of n

(8)

(Total for question = 14 marks)

Q4.

The random variable $X \sim \mathrm{N}(5, 0.4^2)$ and the random variable $Y \sim \mathrm{N}(8, 0.1^2)$

X and Y are independent random variables.

A random sample of a independent observations is taken from the distribution of X and one observation is taken from the distribution of Y

The random variable $W = X_1 + X_2 + X_3 + ... + X_a + bY$ and has the distribution $\mathrm{N}(169, 2^2)$

Find the value of a and the value of b

(Total for question = 6 marks)

Q5.

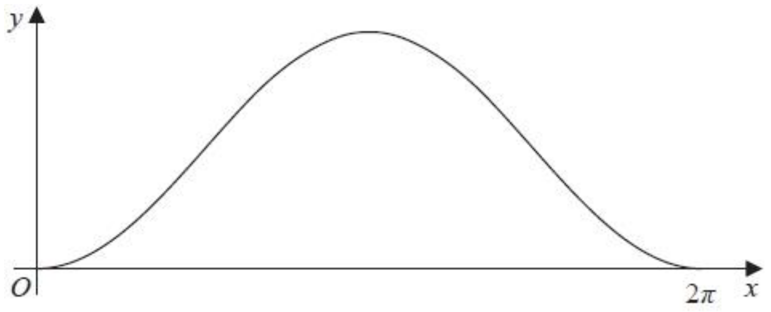

Figure 1

The random variable X has probability density function f (x) and Figure 1 shows a sketch of f (x) where

$$f(x) = \begin{cases} k(1-\cos x) & 0 \leqslant x \leqslant 2\pi \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $k = \dfrac{1}{2\pi}$

(3)

The random variable $Y \sim N(\mu, \sigma^2)$ and $E(Y) = E(X)$

The probability density function of Y is g (y), where

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \qquad -\infty < y < \infty$$

Given that g (μ) = f (μ)

(b) find the exact value of σ

(3)

(c) Calculate the error in using $P\left(\frac{\pi}{2} < Y < \frac{3\pi}{2}\right)$ as an approximation to $P\left(\frac{\pi}{2} < X < \frac{3\pi}{2}\right)$

(4)

(Total for question = 10 marks)

Q6.

Scaffolding poles come in two sizes, long and short. The length L of a long pole has the normal distribution $N(19.6, 0.6^2)$. The length S of a short pole has the normal distribution $N(4.8, 0.3^2)$. The random variables L and S are independent.

A long pole and a short pole are selected at random.

(a) Find the probability that the length of the long pole is more than 4 times the length of the short pole. Show your working clearly.

(6)

Four short poles are selected at random and placed end to end in a row. The random variable T represents the length of the row.

(b) Find the distribution of T.

(3)

(c) Find $P(\,|L - T| < 0.2)$

(4)

(Total for question = 13 marks)

<u>Mark Scheme</u> – Combinations of Random Variables

Q1.

Question Number	Scheme	Notes	Marks
(a)	$L \sim \text{N}(1510, 20^2)$ and $M \sim \text{N}(520, 10^2)$		
	$W = L - (M_1 + M_2 + M_3)$	Allow $L-(M+M+M)$ but **not** $L-3M$ Can be implied by correct $\text{Var}(W)$. May use $W = L-(M_1+M_2+M_3)-15$ for B1.	B1
	$\text{E}(W) = 1510 - 3\times 520 = -50$	Accept 50 if definition reversed. Accept $\text{E}(W) = 1510-3\times 520 - 15 = -65$	B1
	$\text{Var}(W) = 20^2 + 10^2 + 10^2 + 10^2 = 700$	Attempt $\text{Var}(W) = \text{Var}(L) + 3\text{Var}(M)$. Do not condone missing squares, cao.	M1,A1
	$\text{P}(W > 15) \quad = \text{P}\left(Z > \frac{15 - -50}{\sqrt{700}}\right)$	Attempting the correct probability and standardising with their mean and sd dependent on 1st M1. If values for W is not being used or not their variance score M0. Must use 15. Accept $\text{P}(W>0) \quad = \text{P}\left(Z > \frac{0 - -65}{\sqrt{700}}\right)$	dM1
	$= \text{P}(Z > 2.456769\ldots)$		
	$= 0.0069$	0.0071 by calc. awrt 0.007	A1
			(6)
(b)	$X = 3M - L$	Can be implied by correct variance.	
	$\text{E}(X) = 3\times 520 - 1510 = 50$	Accept -50 if reversed.	B1
	$\text{Var}(X) = 3^2 \times 10^2 + 20^2 = 1300$	Attempt $\text{Var}(W) = 3^2\text{Var}(M) + \text{Var}(S)$. Do not condone missing squares, cao. Condone $10^2 + 3^2 \times 20^2$ for M1A0.	M1,A1
	$\text{P}(X > 0) = \text{P}\left(Z > \frac{-50}{\sqrt{1300}}\right)$	Attempting the correct probability and standardising with their mean and sd.	dM1
	$= \text{P}(Z > -1.38675\ldots) = 0.9177$	0.9172 by calc. awrt 0.917-0.918	A1
			(5)
(c)	P(all 5 bags weigh more than 520 grams) = $= \left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$	0.03125	B1

	$\bar{M} \sim \mathrm{N}(520, \frac{10^2}{5})$ or $\sum_{i=1}^{5} M_i \sim \mathrm{N}(2600, 500)$	Both mean and variance required in either case. Can be implied below.	B1
	$\mathrm{P}(\bar{M} > d) = \mathrm{P}\left(Z > \frac{d-520}{\frac{10}{\sqrt{5}}}\right) = 0.03125$ or $\mathrm{P}(T > 5d) = \mathrm{P}\left(Z > \frac{5d-2600}{\sqrt{500}}\right) = 0.03125$	Standardise using d, 520 and 10 or 5d, 2600 and $\sqrt{500}$.	M1
	$\Rightarrow \frac{d-520}{\frac{10}{\sqrt{5}}} = 1.86(27...)$ or $\frac{5d-2600}{\sqrt{500}} = 1.86(27...)$	Equate to z value	M1
	$d = 528.3$	awrt 528.3	A1
			(5)
ALT (c)	Accept use d as difference to 520 provided 520 added to final answer:		
	P(all 5 bags weigh more than 520 grams) = $= \left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$	0.03125	B1
	$\bar{M} \sim \mathrm{N}(0, \frac{10^2}{5})$ or $\sum_{i=1}^{5} M_i \sim \mathrm{N}(0, 500)$	Both mean and variance required in either case. Can be implied below.	B1
	$\mathrm{P}(\bar{M} > d) = \mathrm{P}\left(Z > \frac{d}{\frac{10}{\sqrt{5}}}\right) = 0.03125$ or $\mathrm{P}(T > 5d) = \mathrm{P}\left(Z > \frac{5d}{\sqrt{500}}\right) = 0.03125$	Standardise using d and 10 or $5d$ and $\sqrt{500}$.	M1
	$\Rightarrow \frac{d}{\frac{10}{\sqrt{5}}} = 1.86(27...)$ or $\frac{5d}{\sqrt{500}} = 1.86(27...)$	Equate to z value	M1
	$d = 520 + 8.3 = 528.3$	awrt 528.3	A1
			(5)
			Total 16

Q2.

Question	Scheme	Marks	AOs
(a)	Let $T = W - 2X$ then $\mathrm{E}(T) = 2.5 - 2\times 1.27$	M1	3.3
	$= -0.04$	A1	1.1b
	$\mathrm{Var}(T) = 0.7^2 + 2^2 \times 0.4^2$	M1	2.1
	$= 1.13$	A1	1.1b
	$\mathrm{P}\left(Z > \dfrac{0 - "-0.04"}{\sqrt{"1.13"}}\right) = \mathrm{P}(Z > 0.0376\ldots)$	M1	2.1
	= awrt 0.484/0.485	A1	1.1b
		(6)	
(b)	$B = W_1 + W_2 + \ldots + W_n + X_1 + X_2 + \ldots + X_{2n}$	M1	3.3
	$\mathrm{E}(B) = 5.04n$	B1	1.1b
	$\mathrm{Var}(B) = n \times 0.7^2 + 2n \times 0.4^2$		
	$= 0.81n$	A1	1.1b
	$\pm \dfrac{252 - "5.04n"}{\sqrt{"0.81n"}}$	M1	1.1b
	$\dfrac{252 - "5.04n"}{\sqrt{"0.81n"}} = 0.8$	M1	2.1
	$5.04n + 0.72\sqrt{n} - 252 = 0$ oe		
	$\sqrt{n} = -7.14\ldots$ or 7	M1	1.1b
	$n = 7^2$	M1	1.1b
	$= 49$	A1cso	1.1b
		(8)	
			(14 marks)

Notes:

(a) M1: selecting and using an appropriate model. ie $\pm(W - 2X)$ May be implied by -0.04

A1: -0.04 oe

M1: for realising the need to use $\mathrm{Var}(W) + 4\,\mathrm{Var}(X)$. Allow use of 0.7 for Var (W) instead of 0.7^2 and/or 0.4 for $\mathrm{Var}(X)$ instead of 0.4^2. May be implied by 1.13

A1: 1.13 only

M1: For realising the $\mathrm{P}(T > 0)$ is required and an attempt to find it. $\dfrac{0 - \text{"their } -0.04\text{"}}{\sqrt{\text{"their } 1.13\text{"}}}$ may be implied by a correct answer. If $\mathrm{E}(T)$ and $\mathrm{Var}(T)$ have not been given they must be correct here

A1: awrt 0.484/0.485

(b)M1: Selecting and using appropriate model. May be implied by 0.81

B1: $5.04n$ only

A1: $0.81n$

M1: For standardising using their mean and sd $\pm \dfrac{252 - "5.04n"}{\sqrt{"0.81n"}}$ If mean and sd not given they must be correct here

M1: For constructing an equation and equate their standardisation to 0.8 or awrt 0.7998. Must be of form $\dfrac{252 - an}{b\sqrt{n}} = 0.8$ or $\dfrac{252 - an}{bn} = 0.8$

M1: Correctly solving their 3 term quadratic equation. Condone $n = 7$

M1: for realising the need to square their answer or for attempting to square their quadratic equation

A1cso: 49 only

Q3.

Question	Scheme	Marks	AOs
(a)	$(A+R) \sim \text{N}(300,\ 12^2+10^2)$ $(A_1+A_2) \sim \text{N}(320,\ 2\times12^2)$	M1 A1 A1	3.3 1.1b 1.1b
		(3)	
(b)	P(both are apples) $[=\frac{4}{5}\times\frac{3}{4}]=\frac{3}{5}$ P(one apple and one orange) $=\frac{2}{5}$	M1	2.1
	$'\frac{3}{5}'\text{P}(A_1+A_2>310)+'\frac{2}{5}'\text{P}(A+R>310)$	M1	2.1
	$=0.5377....$ awrt **<u>0.538</u>**	A1	1.1b
		(3)	
(c)	$[W=\sum_1^m A-n\times R]$ $W\sim \text{N}(160m-140n, m\times12^2+n^2\times10^2)$	M1 A1	3.3 1.1b
	$160m - 140n = (1100.08+1499.92)\div2$ [=1300]	M1	2.1
	$2\times1.96\times\sqrt{m\times12^2+n^2\times10^2}=(1499.92-1100.08)$ $[\sqrt{m\times12^2+n^2\times10^2}=102]$	B1 M1	1.1b 1.1b
	$m=\frac{1300+140n}{160}\rightarrow\sqrt{(\frac{1300+140n}{160})\times12^2+n^2\times10^2}=102$ $100n^2+126n-9234=0$	dM1	2.1
	$n=9$ ($n=-10.26$ reject)	A1	1.1b
	$m=16$	A1	1.1b
		(8)	
			(14 marks)

Notes	
(a)	**M1:** Setting up either model for the weights of the two fruit **A1:** Correct distribution for 1 apple 1 orange **A1:** Correct distribution for 2 apples
(b)	**M1:** Finding probability for each possible outcome **M1:** Fully correct method for finding the required probability **A1:** awrt 0.538
(c)	**M1:** Setting up model for W **A1:** correct distribution **M1:** Using given interval to set up equation for mean **B1:** 1.96 **M1:** Using given interval to set up equation for variance **dM1:** Solving simultaneously leading to a 3TQ (dep on previous M mark) **A1:** n = 9 (only) **A1:** m = 16 (only)

Q4.

Question	Scheme	Marks	AOs
	$5a+8b=169$	B1	1.1b
	$0.16a+0.01b^2=4$	B1	1.1b
	$a=33.8-1.6b \rightarrow 0.16(33.8-1.6b)+0.01b^2=4$	M1	2.1
	$0.01b^2-0.256b+1.408=0 \rightarrow b=\dfrac{0.256\pm\sqrt{0.256^2-4(0.01)(1.408)}}{2(0.01)}$	M1	1.1b
	$b=8,\ a=21$ (reject $b=17.6,\ a=5.64$ since $a\in\mathbb{Z}^+$)	A1A1	1.1b 2.2a
		(6 marks)	
	Notes		
	B1: Correct equation for the means **B1:** Correct equation for the variances (allow $0.4^2a+0.1^2b^2=4$) **M1:** Attempt to solve two simultaneous equations in a and b by eliminating one variable **M1:** Attempt to solve their quadratic (must be seen if answers are incorrect) **A1:** $b=8$ or $a=21$ or both sets of values of a and b without rejecting **A1:** Choosing correct pair of solutions $b=8,\ a=21$ only		

Q5.

Qu	Scheme	Marks	AO
(a)	$\int(1-\cos x)\ \mathrm{d}x=[x-\sin x]$	M1	1.1b
	Use of correct limits and $\int \mathrm{f}(x)\ \mathrm{d}x=1 \Rightarrow 2\pi-0-0\ =1$	M1	1.1b
	so $k=\frac{1}{2\pi}$ (*)	A1*cso	1.1b
		(3)	
(b)	$\mathrm{E}(X)=\pi$ (symmetry) so $\mu=\pi$ so $\mathrm{f}(\mu)=\dfrac{1}{2\pi}(1-\cos\pi)=\dfrac{1}{\pi}$	B1	2.2a
	$\dfrac{1}{\sigma\sqrt{2\pi}}=\text{"}\dfrac{1}{\pi}\text{"}$; so $\sigma=\sqrt{\dfrac{\pi}{2}}$	M1; A1	1.1b 1.1b
		(3)	
(c)	$\mathrm{P}\left(\dfrac{\pi}{2}<X<\dfrac{3\pi}{2}\right)=\dfrac{1}{2\pi}[x-\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=\dfrac{1}{2\pi}\left[\left(\dfrac{3\pi}{2}--1\right)-\left(\dfrac{\pi}{2}-1\right)\right]$	M1	3.4
	$=\dfrac{2+\pi}{2\pi}$ $(=0.81830\ldots)$	A1	1.1b
	$\mathrm{P}\left(\dfrac{\pi}{2}<Y<\dfrac{3\pi}{2}\right)=0.7899\ldots$	B1	1.1b
	So error is $0.81830\ldots-0.7899\ldots=0.0284$	A1 (4)	1.1b
		(10 marks)	
	Notes		
(a)	1st M1 attempt to integrate $(1-\cos x)$ – one correct term 2nd M1 for use of correct limits and correct method for k A1* cso use of $\int \mathrm{f}(x)\ \mathrm{d}x=1$ seen and no incorrect working seen		
(b)	B1 for correctly deducing the value of $\mathrm{f}(\mu)$ M1 for a correct equation for σ – ft their value for $\mathrm{f}(\mu)$ [condone for sight of correct $\mathrm{g}(\mu)$] A1 for $\sqrt{\frac{\pi}{2}}$ or exact equivalent		
(c)	M1 for a correct attempt to find prob – some correct integration and use of limits 1st A1 for a correct answer (exact or 0.818.. or better) B1 for a correct probability from their calculator i.e. 0.7899 or better accept 0.79 2nd A1 for 0.0284 or better		

Q6.

Question	Scheme	Marks	AOs
(a)	Let $X = L - 4S$ then $\text{E}(X) = 19.6 - 4\times 4.8$	M1	2.3
	$= 0.4$	A1	1.1b
	$\text{Var}(X) = \text{Var}(L) + 4^2\text{Var}(S) = 0.6^2 + 16\times 0.3^2$	M1	2.1
	$= 1.8$	A1	1.1b
	$\text{P}(X > 0) = [\text{P}(Z > \dfrac{0-0.4}{\sqrt{1.8}} = -0.298\ldots\ldots)]$	M1	2.1
	$= 0.617202..$ awrt **<u>0.617</u>**	A1	1.1b
		(6)	
(b)	$T = S_1 + S_2 + S_3 + S_4$ (May be implied by 0.36)	M1	3.3
	$T \sim \text{N}(19.2, 0.36)$ $\text{E}(T) = 19.2$	B1	1.1b
	$\text{Var}(T) = 0.36$ or 0.6^2	A1	1.1b
		(3)	
(c)	Let $Y = L - T$ $\text{E}(Y) = \text{E}(L) - \text{E}(T) = [\,0.4]$	M1	3.3
	$\text{Var}(Y) = \text{Var}(L) + \text{Var}(T) = [\,0.72]$	M1	1.1b
	Require $\text{P}(-0.2 < Y < 0.2)$	M1	3.1a
	$= 0.16708\ldots$ awrt **<u>0.167</u>**	A1	1.1b
		(4)	
			(13 marks)

Notes	
(a)	**M1:** Selecting and using an appropriate model i.e $\pm(L - 4S)$. May be implied by 0.4 **A1:** 0.4 oe **M1:** For realising the need to use $\text{Var}(L) + 4^2\text{Var}(S)$. Allow use of 0.6 for $\text{Var}(L)$ instead of 0.6^2 and/or 0.3 for $\text{Var}(S)$ instead of 0.3^2 may be implied by 1.8 **A1:** 1.8 only **M1:** For realising $\text{P}(X > 0)$ is required and an attempt to find it e.g. $\dfrac{0-0.4}{\sqrt{\text{"their Var}(X)\text{"}}}$ but do not allow a negative $\text{Var}(X)$ **A1:** awrt 0.617
(b)	**M1:** Selecting and using an appropriate model ie $S_1 + S_2 + S_3 + S_4$: may be implied by 0.36 **B1:** 19.2 only **A1:** 0.36
(c)	**M1:** Setting up and using the model $Y = L - T$. May be implied by $E(Y) = E(L) - E(T)$ **M1:** Using $\text{Var}(Y) = \text{Var}(L) + \text{Var}(T)$ **M1:** Dealing with the modulus and realising they need to find $\text{P}(-0.2 < Y < 0.2)$ **A1:** awrt 0.167