Series

Questions

Q1.

In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2 (2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

(Total for question = 9 marks)

Q2.

(a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n} (5r - 2)^2 = \frac{1}{6} n(an^2 + bn + c)$$

where a, b and c are integers to be determined.

(5)

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k} (5r - 2)^2 = 94k^2$$

(4)

(Total for question = 9 marks)

Q3.

(a) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers n,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c and d are integers to be determined.

(3)

(Total for question = 9 marks)

Q4.

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (3r^2 + 8r + 3) = \frac{1}{2}n(2n+5)(n+3)$$

for all positive integers n.

(5)

Given that

$$\sum_{r=1}^{12} \left(3r^2 + 8r + 3 + k(2^{r-1})\right) = 3520$$

(b) find the exact value of the constant k.

(4)

(Total for question = 9 marks)

Q5.

(a) Using the formula for $\sum_{r=1}^{n} r^2$ write down, in terms of *n* only, an expression for

$$\sum_{r=1}^{3n} r^2$$

(1)

(b) Show that, for all integers *n*, where *n* > 0

$$\sum_{r=2n+1}^{3n} r^2 = \frac{n}{6}(an^2 + bn + c)$$

where the values of the constants *a*, *b* and *c* are to be found.

(4)

(Total for question = 5 marks)

Q6.

(a) Prove by induction that for all positive integers *n*,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers *n*,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of *n* that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

(Total for question = 15 marks)

Mark Scheme - Series

Q1.

Question	Scheme	Marks	AOs
(a)	A complete attempt to find the sum of the cubes of the first n odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1	3.1a
	$\sum_{r=1}^{n}(2r-1)^3=\sum_{r=1}^{n}(8r^3-12r^2+6r-1)=8\sum_{r=1}^{n}r^3-12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3=\sum_{r=0}^{n-1}(8r^3+12r^2+6r+1)=8\sum_{r=0}^{n-1}r^3+12\sum_{r=0}^{n-1}r^2+6\sum_{r=0}^{n-1}r+\sum_{r=0}^{n-1}1$	M1	1.1b
	$=8\frac{n^2}{4}(n+1)^2-12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)-n$ or $=8\frac{(n-1)^2}{4}(n)^2+12\frac{(n-1)}{6}(n)(2n-1)+6\frac{(n-1)}{2}(n)+n$	M1 A1	1.1b 1.1b
	Multiplies out to achieve a correct intermediate line for example $n\ n+1\ \ 2n^2-2n+1\ -n=2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $=n^2(2n^2-1)$ cso *	A1 *	2.1
		(5)	
(b)	$\sum_{r=n}^{n+9}(2r-1)^3=\sum_{r=1}^{n+9}(2r-1)^3-\sum_{r=1}^{n-1}(2r-1)^3$ $=(n+9)^2\left(2(n+9)^2-1\right)-(n-1)^2\left(2(n-1)^2-1\right)=99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3=\sum_{r=1}^{n+10}(2r-1)^3-\sum_{r=1}^{n}(2r-1)^3$ $=(n+10)^2\left(2(n+10)^2-1\right)-(n)^2\left(2n^2-1\right)=99800$ or $\sum_{r=n-9}^{n}(2r-1)^3=\sum_{r=1}^{n}(2r-1)^3-\sum_{r=1}^{n-10}(2r-1)^3$ $=(n)^2\left(2(n)^2-1\right)-(n-10)^2\left(2(n-10)^2-1\right)=99800$	M1	3.1a
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1	1.1b
	Solves cubic equation	dM1	1.1b

	Achieves $n=6$ and the smallest number as 11 or Achieves $n=5$ and the smallest number as 11 or Achieves $n=15$ and the smallest number as 11	A1	2.3
		(4)	
			(9 marks)

Notes:

(a)

M1: A complete attempt to find the sum of the cubes of n odd numbers using three of the standard summation formulae.

M1: Expands $\sum_{r=1}^{n}(2r-1)^3$ or $\sum_{r=0}^{n-1}(2r+1)^3$ and splits into fours appropriate sums.

M1: Applies the result for at least three summations $\sum_{r=0}^{n-1}r^3, \sum_{r=0}^{n-1}r^2, \sum_{r=0}^{n-1}r$ and $\sum_{r=0}^{n-1}1$ or

$\sum_{r=1}^{n}r^3, \sum_{r=1}^{n}r^2, \sum_{r=1}^{n}r$ and $\sum_{r=1}^{n}1$ as appropriate to their expansion provided that there is an attempt at cubing some values.

A1: Correct unsimplified expression.

A1 *: Multiplies out to achieve a correct intermediate expression which clearly leads to the correct expression. cso

Special case: If uses $\sum_{r=1}^{n}(2r+1)^3$ leading to $=8\frac{n^2}{4}(n+1)^2+12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)+n$ max score is M1 M0 M1 A1 A0

(b)

M1: Uses the answer to part (a) to find the sum of the cubes of the first $N+10$ odd numbers minus the sum of the first N odd numbers and sets equal to 99800 or equivalent.

A1: Correct simplified cubic equation.

dM1: Uses their calculator to solve their cubic equation, dependent on previous method mark.

A1: cao

Q2.

Question	Scheme	Marks	AOs
(a)	$(5r-2)^2 = 25r^2 - 20r + 4$	B1	1.1b
	$\sum_{r=1}^{n} 25r^2 - 20r + 4 = \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + ...$	M1	2.1
	$= \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + 4n$	A1	1.1b
	$= \frac{1}{6}n\left[25(2n^2+3n+1) - 60(n+1) + 24\right]$	dM1	1.1b
	$= \frac{1}{6}n\left[50n^2 + 15n - 11\right]$	A1	1.1b
		(5)	
(b)	$\frac{1}{6}k\left[50k^2 + 15k - 11\right] = 94k^2$	M1	1.1b
	$50k^3 - 549k^2 - 11k = 0$ or $50k^2 - 549k - 11 = 0$	A1	1.1b
	$(k-11)(50k+1) = 0 \Rightarrow k = ...$	M1	1.1b
	$k = 11$(only)	A1	2.3
		(4)	
			(9 marks)

Notes
(a) B1: Correct expansion M1: Substitutes at least one of the standard formulae into their expanded expression A1: Fully correct expression dM1: Attempts to factorise $\frac{1}{6}n$ having used at least one standard formula correctly. Dependent on the first M mark. A1: Obtains the correct expression or the correct values of a, b and c (b) M1: Uses their result from part (a) and sets equal to $94k^2$ and attempt to expand and collect terms. A1: Correct cubic or quadratic M1: Attempts to solve their 3TQ or cubic equation A1: Identifies the correct value of k with no other values offered

Q3.

Question	Scheme	Marks	AOs
(a)	$n=1,\quad \text{lhs}=1(2)(3)=6,\quad \text{rhs}=\frac{1}{2}(1)(2)^2(3)=6$ (true for $n=1$)	B1	2.2a
	Assume true for $n=k$ so $\sum_{r=1}^{k} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)$	M1	2.4
	$\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$	M1	2.1
	$=\frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$	dM1	1.1b
	$=\frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$ Shows that $=\frac{1}{2}(\underline{k+1})(\underline{k+1}+1)^2(\underline{k+1}+2)$ Alternatively shows that $\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}(k+1)(k+1+1)^2(k+1+2)$ $=\frac{1}{2}(k+1)(k+2)^2(k+3)$ Compares with their summation and concludes true for $n=k+1$, may be seen in the conclusion.	A1	1.1b
	If the statement is **true for** $n=k$ **then** it has been shown **true for** $n=k+1$ and as it is **true for** $n=1$, the statement is **true** for all **positive integers** n.	A1	2.4
		(6)	
(b)	$\sum_{r=n}^{2n} r(r+1)(2r+1)=\frac{1}{2}(2n)(2n+1)^2(2n+2)-\frac{1}{2}(n-1)n^2(n+1)$	M1	3.1a
	$=\frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right]$	M1	1.1b
	$=\frac{1}{2}n(n+1)(15n^2+17n+4)$ $=\frac{1}{2}n(n+1)(3n+1)(5n+4)$	A1	1.1b
		(3)	
			(9 marks)

Notes
(a) Note ePen B1 M1 M1 A1 A1 A1 B1: Substitutes $n = 1$ into both sides to show that they are both equal to 6. (There is no need to state true for $n = 1$ for this mark) M1: Makes a statement that assumes the result is true for some value of n, say k M1: Adds the $(k + 1)$th term to the assumed result dM1: Dependent on previous M, factorises out $\frac{1}{2}(k+1)(k+2)$ A1: Reaches a correct the required expression no errors and shows that this is the correct sum for $n = k+1$ A1: Depends on all except **B** mark being scored (must have been some attempt to show true for $n = 1$). Correct conclusion conveying all the points in bold. (b) M1: Realises that $\sum_{r=1}^{2n} r(r+1)(2r+1) - \sum_{r=1}^{n-1} r(r+1)(2r+1)$ is required and uses the result from part (a) to obtain the required sum in terms of n M1: Attempts to factorise by $\frac{1}{2}n(n+1)$ A1: Correct expression or correct values

Q4.

Question Number	Scheme		Marks
(a)	$\sum_{r=1}^{n}\left(3r^2+8r+3\right)$		
	$= \frac{3}{6}n(n+1)(2n+1)+\frac{8}{2}n(n+1)+3n$	An attempt to use at least one of the correct standard formulae for first two terms.	M1
		Correct first two terms.	A1
		$3 \to 3n$	B1
	$= \frac{1}{2}n(n+1)(2n+1)+4n(n+1)+3n$		
	$= \frac{1}{2}n\left((2n+1)(n+1)+8(n+1)+6\right)$	Factorise out at least n from all terms at any point. There must be a factor of n in every term.	M1
	$= \frac{1}{2}n\left(2n^2+3n+1+8n+8+6\right)$		
	$= \frac{1}{2}n\left(2n^2+11n+15\right)$		
	$= \frac{1}{2}n(2n+5)(n+3)$ (*)	Achieves the correct answer, no errors seen.	A1*cso
			[5]
	$\sum_{r=1}^{12}\left(3r^2+8r+3+k(2^{r-1})\right) = 3520$		
(b)	$\sum_{r=1}^{12}\left(3r^2+8r+3\right) = \frac{1}{2}(12)(29)(15)\ \{=2610\}$	Attempt to evaluate $\sum_{r=1}^{12}\left(3r^2+8r+3\right)$	M1
	$\sum_{r=1}^{12}(2^{r-1}) = \frac{1(1-2^{12})}{1-2}\ \{=4095\}$	Attempt to apply the sum to 12 terms of a GP or adds up all 12 terms.	M1
		$\frac{1(1-2^{12})}{1-2}$ o.e. or 4095.	A1
	So, $2610+4095k=3520 \Rightarrow 4095k=910$		
	giving, $k=\frac{2}{9}$	$k=\frac{2}{9}$ or $0.\dot{2}$	A1
			[4] **9**

	Question Notes
(b)	**Note** 2^{nd} M1 1^{st} A1: These two marks can be implied by seeing 4095 or $4095k$

Q5.

Question Number	Scheme	Marks
	(a) $\sum_{r=1}^{3n} r^2 = \frac{1}{6}3n(3n+1)(6n+1)$ or $\sum_{r=1}^{3n} r^2 = \frac{1}{2}n(3n+1)(6n+1)$ or equivalent	B1 (1)
	(b) See $\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$ or equivalent	B1
	Attempt to use $\sum_{r=1}^{3n} r^2 - \sum_{r=1}^{2n} r^2 = \frac{n}{6}\{3(3n+1)(6n+1) - 2(2n+1)(4n+1)\}$	M1
	$= \frac{n}{6}\{(54n^2+27n+3)-(16n^2+12n+2)\}$	dM1
	$= \frac{n}{6}\{(38n^2+15n+1)\}$	A1
	$(a = 38, b = 15, c = 1)$	
		(4)
		5 marks

Notes

(a) B1: Either right hand side or exact equivalent - isw if expanded

(b) B1: States or uses $\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$

M1: Subtracts their sum to $2n$ or $2n-1$ **and** attempts to factorise by $\frac{n}{6}$ seen anywhere.

dM1: Expands two quadratics dependent on first M1

A1: cao

Q6.

Question	Scheme	Marks	AOs
(a)	$n=1,\ \sum_{r=1}^{1} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1)(2)(3) = 1$	B1	2.2a
	Assume general statement is true for $n = k$. So assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true.	M1	2.4
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	2.1
	$= \frac{1}{6}(k+1)(2k^2+7k+6)$	A1	1.1b
	$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$	A1	1.1b
	Then the general result is true for $n = k+1$. As the general result has been shown to be true for $n = 1$, then the general result is true for all $n \in \mathbb{Z}^+$.	A1	2.4
		(6)	
(b)	$\sum_{r=1}^{n} r(r+6)(r-6) = \sum_{r=1}^{n} (r^3 - 36r)$		
	$= \frac{1}{4}n^2(n+1)^2 - \frac{36}{2}n(n+1)$	M1 A1	2.1 1.1b
	$= \frac{1}{4}n(n+1)\left[n(n+1) - 72\right]$	M1	1.1b
	$= \frac{1}{4}n(n+1)(n-8)(n+9)$ * **cso**	A1*	1.1b
		(4)	
(c)	$\frac{1}{4}n(n+1)(n-8)(n+9) = \frac{17}{6}n(n+1)(2n+1)$	M1	1.1b
	$\frac{1}{4}(n-8)(n+9) = \frac{17}{6}(2n+1)$	M1	1.1b
	$3n^2 - 65n - 250 = 0$	A1	1.1b
	$(3n+10)(n-25) = 0$	M1	1.1b
	(As n must be a positive integer,) $n = 25$	A1	2.3
		(5)	
			(15 marks)

		Question Notes
(a)	B1	Checks $n = 1$ works for both sides of the general statement.
	M1	Assumes (general result) true for $n = k$.
	M1	Attempts to add $(k+1)$th term to the sum of k terms.
	A1	Correct algebraic work leading to either $\frac{1}{6}(k+1)(2k^2+7k+6)$ or $\frac{1}{6}(k+2)(2k^2+5k+3)$ or $\frac{1}{6}(2k+3)(k^2+3k+2)$
	A1	Correct algebraic work leading to $\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$
	A1	cso leading to a correct induction statement conveying **all** three underlined points.
(b)	M1	Substitutes at least one of the standard formulae into their expanded expression.
	A1	Correct expression.
	M1	Depends on previous M mark. Attempt to factorise at least $n(n+1)$ having used both standard formulae correctly.
	A1*	Obtains $\frac{1}{4}n(n+1)(n-8)(n+9)$ by **cso**.
(c)	M1	Sets their part (a) answer equal to $\frac{17}{6}n(n+1)(2n+1)$
	M1	Cancels out $n(n+1)$ from both sides of their equation.
	A1	$3n^2 - 65n - 250 = 0$
	M1	A valid method for solving a 3 term quadratic equation.
	A1	**Only one solution** of $n = 25$