# Conic Sections 2

## Questions

**Q1.**

The ellipse $E$ has equation

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

The line $l$ is the normal to $E$ at the point $P$ $(6\cos\theta, 5\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$

(a) Use calculus to show that an equation of $l$ is

$$6x\sin\theta - 5y\cos\theta = 11\sin\theta\cos\theta$$

**(5)**

The line $l$ meets the $x$-axis at the point $Q$.

The point $R$ is the foot of the perpendicular from $P$ to the $x$-axis.

(b) Show that $\frac{OQ}{OR} = e^2$, where $e$ is the eccentricity of the ellipse $E$.

**(4)**

**(Total for question = 9 marks)**

**Q2.**

The ellipse $E$ has equation

$$\frac{x^2}{36} + \frac{y^2}{20} = 1$$

Find

(a) the coordinates of the foci of $E$,

**(3)**

(b) the equations of the directrices of $E$.

**(2)**

**(Total for question = 5 marks)**

**Q3.**

The parabola $C$ has equation

$$y^2 = 32x$$

and the hyperbola $H$ has equation

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

(a) Write down the equations of the asymptotes of $H$.

**(1)**

The line $l_1$ is normal to $C$ and parallel to the asymptote of $H$ with positive gradient.

The line $l_2$ is normal to $C$ and parallel to the asymptote of $H$ with negative gradient.

(b) Determine

(i) an equation for $l_1$
(ii) an equation for $l_2$

**(4)**

The lines $l_1$ and $l_2$ meet $H$ at the points $P$ and $Q$ respectively.

(c) Find the area of the triangle $OPQ$, where $O$ is the origin.

**(4)**

**(Total for question = 9 marks)**

**Q4.**

An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and eccentricity $e_1$

A hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity $e_2$

Given that $e_1 \times e_2 = 1$

(a) show that $a^2 = 3b^2$

**(4)**

Given also that the coordinates of the foci of the ellipse are the same as the coordinates of the foci of the hyperbola,

(b) determine the equation of the hyperbola.

**(3)**

**(Total for question = 7 marks)**

Q5.

The hyperbola $H$ has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The line $l_1$ is the tangent to $H$ at the point $P(4\cosh\theta, 3\sinh\theta)$.

The line $l_1$ meets the $x$-axis at the point $A$.

The line $l_2$ is the tangent to $H$ at the point (4, 0).

The lines $l_1$ and $l_2$ meet at the point $B$ and the midpoint of $AB$ is the point $M$.

(a) Show that, as $\theta$ varies, a Cartesian equation for the locus of $M$ is

$$y^2 = \frac{9(4-x)}{4x} \qquad p < x < q$$

where $p$ and $q$ are values to be determined.

**(11)**

Let $S$ be the focus of $H$ that lies on the positive $x$-axis.

(b) Show that the distance from $M$ to $S$ is greater than 1

**(3)**

**(Total for question = 14 marks)**

# Mark Scheme – Conic Sections 2

**Q1.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| (a) | $\frac{2x}{36}+\frac{2y}{25}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{25x}{36y}=\frac{5\cos\theta}{-6\sin\theta}$ or <br> $x=6\cos\theta, y=5\sin\theta\Rightarrow\frac{dy}{dx}=\frac{5\cos\theta}{-6\sin\theta}$ or <br> $\frac{y^2}{25}=1-\frac{x^2}{36}\Rightarrow y=5\sqrt{1-\frac{x^2}{36}}\Rightarrow\frac{dy}{dx}=-\frac{5x}{36}\left(1-\frac{x^2}{36}\right)^{-\frac{1}{2}}=-\frac{5\cos\theta}{6\sin\theta}$ <br> M1: Correct attempt at $\frac{dy}{dx}$ using implicit or parametric or explicit differentiation <br> $\left(ax+by\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=...,\frac{dy}{dx}=\pm\frac{a\cos\theta}{b\sin\theta},\frac{dy}{dx}=ax\left(1-bx^2\right)^{-\frac{1}{2}}(oe)\Rightarrow\frac{dy}{dx}=...\right)$ | | M1 |
| | $=-\frac{5\cos\theta}{6\sin\theta}$ | A1: Correct tangent gradient in terms of $\theta$. May be implied in their attempt the normal gradient. | A1 |
| | $m_N=\frac{6\sin\theta}{5\cos\theta}$ | Correct perpendicular gradient rule. May be awarded if working in terms of $x$ and $y$. | M1 |
| | $y-5\sin\theta=\mathit{Their}\ m_N\left(x-6\cos\theta\right)$ | Correct straight line method for the normal using a "changed" $\frac{dy}{dx}$ in terms of θ which must have come from calculus. If using $y=mx+c$, must reach as far as $c=\ldots$ | M1 |
| | $6x\sin\theta-5y\cos\theta=11\sin\theta\cos\theta$* | Correct completion to printed answer with no errors. | A1* |
| | Note that if the candidate uses e.g $y-5\sin\theta=-\frac{36y}{25x}\left(x-6\cos\theta\right)$ before introducing $\theta$, the final mark can be withheld. | | |
| | | | **(5)** |
| (b) | $b^2=a^2\left(1-e^2\right)\Rightarrow 25=36\left(1-e^2\right)\Rightarrow e^2=\frac{11}{36}$ <br> or $e=\sqrt{\frac{11}{36}}$ | Uses the correct eccentricity formula to obtain a value for $e$ or $e^2$. Ignore ± values for e. | M1 |
| | $y=0\Rightarrow x=\frac{11\cos\theta}{6}$ or $\frac{11\sin\theta\cos\theta}{6\sin\theta}$ | Correct $x$ coordinate for $Q$ | B1 |
| | $\left(\frac{OQ}{OR}=\right)\frac{11\cos\theta}{6}\times\frac{1}{6\cos\theta}$ | Attempts $\frac{\text{their } OQ}{\text{their } OR}$. May be implied by their ratio. | M1 |
| | $=\frac{11}{36}$ | Correct completion with no errors to obtain $\frac{11}{36}$ both times. | A1 |
| | Ignore any references to the foci or directrices but the final mark can be withheld if there are any incorrect statements such as e.g. using cos θ = 1 in their ratio. | | |
| | | | **(4)** |
| | | | **Total 9** |

**Q2.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Uses $b^2 = a^2\left(1-e^2\right)$ to find a value of $e$ look for $20 = 36\left(1-e^2\right)$ | M1 | 1.1b |
| | $e = \frac{2}{3} \Rightarrow$ foci are $\left(\pm 6 \times \text{"their } e\text{"}, 0\right)$ | dM1 | 1.1b |
| | Foci are $(\pm 4, 0)$ | A1 | 1.1b |
| | | (3) | |
| | **Alternative**<br>Sets up an equation such as $2\sqrt{p^2 + b^2} = 2a$ where $p$ is the $x$ coordinate of the foci $2\sqrt{p^2 + 20} = 12$ | M1 | 1.1b |
| | Solves to find the value of $p$ | dM1 | 1.1b |
| | Foci are $(\pm 4, 0)$ | A1 | 1.1b |
| | | (3) | |
| (b) | Directrices are $x = (\pm)\frac{6}{\text{their } e}$ | M1 | 1.1b |
| | $x = \pm 9$ only | A1 | 1.1b |
| | | (2) | |
| | | **(5 marks)** | |

**Notes:**

**(a)**

**M1**: Uses $b^2 = a^2(1-e^2)$ to obtain a value of $e$ (allow if $-\frac{2}{3}$ also given)

**dM1**: Uses $a = 6$ and their value of $e$ with $0 < e < 1$, to find at least one focus using $\left((\pm)ae, 0\right)$

**A1**: Correct foci – both required, including $y$ coordinates.

**Alternative**

**M1**: Sets up an equation using total distance from foci to point on ellipse = 2a

**dM1**: Solves to find a value for the $x$ coordinate of the foci

**A1**: Correct foci – both required, including $y$ coordinates.

**(b)**

**M1**: Uses $x = (\pm)\frac{a}{e}$ with $a = 6$ and their $e$ to attempt directrices.

**A1**: Correct directrices, both required and no other lines

**Q3.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Equations of asymptotes of $H$ are $y=\pm\frac{1}{2}x$ oe<br>e.g $y=\pm\frac{3}{6}x$ or $\frac{x}{6}=\pm\frac{y}{3}$ | B1 | 1.1b |
| | | (1) | |
| (b) | For parabola $2y\frac{dy}{dx}=32\Rightarrow\frac{dy}{dx}=...$ or $y=\sqrt{32x}\Rightarrow\frac{dy}{dx}=...x^{-\frac{1}{2}}$ or<br>$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=...$ | M1 | 2.1 |
| | Finds the gradient of the normal using $m_N=\frac{-1}{\text{their}\frac{dy}{dx}}$<br>$m_N=-\frac{y}{16}$ or $-t$ or $-\frac{\sqrt{x}}{2\sqrt{2}}$ so $m_N=(\pm)\frac{1}{2}\Rightarrow y=(\pm)8, x=2$ | M1 | 3.1a |
| | Finds the equation of either $l_1$ or $l_2$<br>$y-"8"="\text{their } m_N"(x-"2")$ or $y-"-8"="\text{their } m_N"(x-"2")$ | M1 | 1.1b |
| | $l_1$ is $y+8=\frac{1}{2}(x-2)$ and $l_2$ is $y-8=-\frac{1}{2}(x-2)$ oe<br>$y=\frac{1}{2}x-9$ and $y=-\frac{1}{2}x+9$ | A1 | 1.1b |
| | | (4) | |

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|---|---|---|---|
| (c) | Meet $H\Rightarrow\frac{x^2}{36}-\frac{\left(\pm\left(\frac{1}{2}x-9\right)\right)^2}{9}=1\Rightarrow\frac{x^2}{36}-\frac{\frac{1}{4}x^2-9x+81}{9}=1\Rightarrow x=...$<br>or $\frac{(18\pm2y)^2}{36}-\frac{y^2}{9}=1\Rightarrow\frac{81\pm18y+y^2}{9}-\frac{y^2}{9}=1\Rightarrow y=...$ | M1 | 2.1 |
| | One correct point of intersection $(10,\pm4)$ | A1 | 2.2a |
| | Area $OPQ$ is $\frac{1}{2}\times10\times(4-(-4))=...$<br>$-\frac{1}{2}\begin{vmatrix}10 & 4 & 0\\ 10 & -4 & 0\end{vmatrix}=-\frac{1}{2}\left[40-40\right]$<br>$-\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 10 & 4 & 1\\ 10 & -4 & 1\end{vmatrix}=-\frac{1}{2}\left[0-0+\left((10\times-4)-(10\times4)\right)\right]$ | dM1 | 1.1b |
| | $=40$ | A1 | 1.1b |
| | | (4) | |

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|---|
| **(9 marks)** |
| **Notes:** |
| (a)<br>**B1**: Correct equations for the asymptotes of $H$ seen or implied, any form and need not be simplified.<br>**(b) Note M1 M1 A1 A1 on ePen**<br>**M1**: A correct method to find the gradient of the parabola.<br>**M1**: Finds the gradient of the normal and sets their normal gradient equal to their asymptote gradient to obtain at least one point on $C$ where normal is parallel to an asymptote<br>**M1**: Finds the equation of either $l_1$ or $l_2$<br>**A1**: Correct equation for each normal, $y+8=\frac{1}{2}(x-2)$ and $y-8=-\frac{1}{2}(x-2)$. Ignore labelling.<br>(c)<br>**M1**: Substitutes for $x$ or $y$ into the equation of the hyperbola and solves for their variable.<br>**A1**: Achieves one correct coordinate $x=10$ and $y=\pm 4$<br>**dM1**: Dependent on previous method. Correct method for the area of their triangle e.g,<br>$\frac{1}{2}\times$ their $10\times$ twice their 4 or equivalent determinant methods.<br>**A1**: Area is 40. Correct answer only. |

**Q4.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $b^2 = a^2(1-e_1^2) \Rightarrow 4 = 16(1-e_1^2) \Rightarrow e_1^2 = \ldots$ | M1 | 1.1b |
| | $e_1^2 = \frac{3}{4}$ or $e_1 = \frac{\sqrt{3}}{2}$ | A1 | 1.1b |
| | E.g. $b^2 = a^2(e_2^2 - 1) = a^2\left(\frac{1}{e_1^2} - 1\right) = a^2\left(\frac{4}{3} - 1\right)$ | dM1 | 2.1 |
| | $\Rightarrow b^2 = \frac{1}{3}a^2 \Rightarrow a^2 = 3b^2$ * cso | A1* | 1.1b |
| | | **(4)** | |
| (b) | For the focus of the ellipse $(x =)4 \times '\frac{\sqrt{3}}{2}'$ | M1 | 1.1b |
| | For focus of the hyperbola $(x =)\ a \times '\frac{2}{\sqrt{3}}' \Rightarrow 2\sqrt{3} = \frac{2a}{\sqrt{3}} \Rightarrow a = \ldots (= 3)$<br>$\Rightarrow b^2 = \frac{1}{3}a^2 = \ldots$ | M1 | 3.1a |
| | $\frac{x^2}{9} - \frac{y^2}{3} = 1$ cso | A1 | 2.2a |
| | | **(3)** | |
| | | **(7 marks)** | |

**Notes:**

**(a)**

**M1:** Uses "$b^2 = a^2(1-e_1^2)$" with values for $a$ and $b$ to find a value for $e_1$ or $e_1^2$. They may just call it $e$ and will likely use $a$ and $b$ before substituted, which is fine. The formula must be correct but allow slips with $a$ and $b$.

**A1:** Correct exact value for $e_1$ or $e_1^2$. Note: allow M1A1 here if the relevant work is seen in (b).

**dM1:** Dependent on previous method mark. Uses $e_1 \times e_2 = 1$ with their $e_1$ or $e_1^2$ to find an expression between $a$ and $b$. May find an expression for $e_2{}^{(2)}$ and apply $e_1 \times e_2 = 1$ directly or may first substitute as per scheme. Any full method.

SC: Allow M0A0dM1A0 if $b^2 = a^2(1-e_1)$ and $b^2 = a^2(e_2 - 1)$ are used in an otherwise correct process.

**A1*:** Achieves $a^2 = 3b^2$ with at least one intermediate unsimplified equation in $a$ and $b$ cso

**(b)**

**M1:** Uses/implies $x$ coordinate of focus for the ellipse is $4 \times$ their $e_1$

**M1:** For a full process to find values for $a$ and $b$ or their squares. E.g. for focus of hyperbola $x = a \times$ their $e_2 = \frac{a}{e_1}$ sets equal to $4e_1$ and solves for $a$ then attempting to use $a^2 = 3b^2$ to obtain $b^2$(or $b$). Other methods are possible.

**A1:** Deduces the correct equation for the hyperbola.

**Q5.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $\frac{x^2}{16}-\frac{y^2}{9}=1\Rightarrow\frac{x}{8}-\frac{2yy'}{9}=0\Rightarrow y'=\frac{9x}{16y}=\frac{36\cosh\theta}{48\sinh\theta}$ or $x=4\cosh\theta, y=3\sinh\theta\Rightarrow\frac{dy}{dx}=\frac{3\cosh\theta}{4\sinh\theta}$ | M1 | 3.1a |
| | $y-3\sinh\theta=\frac{3\cosh\theta}{4\sinh\theta}(x-4\cosh\theta)$ | M1 | 3.1a |
| | $y=0\Rightarrow x=\frac{4}{\cosh\theta}$ | A1 | 2.2a |
| | line $l_2$ has equation $x=4$ | B1 | 2.2a |
| | $x=4\Rightarrow y-3\sinh\theta=\frac{3\cosh\theta}{4\sinh\theta}(4-4\cosh\theta)$ | M1 | 2.1 |
| | $y=\frac{3\cosh\theta-3}{\sinh\theta}$ | A1 | 2.2a |
| | $M$ is $\left(\frac{1}{2}\left(4+\frac{4}{\cosh\theta}\right),\frac{1}{2}\left(\frac{3\cosh\theta-3}{\sinh\theta}\right)\right)$ | M1 | 1.1b |
| | $x=2+\frac{2}{\cosh\theta}\Rightarrow\cosh\theta=\frac{2}{x-2}$ $\Rightarrow y^2=\frac{9(\cosh\theta-1)^2}{4\sinh^2\theta}=\frac{9\left(\frac{2}{x-2}-1\right)^2}{4\left(\left(\frac{2}{x-2}\right)^2-1\right)}$ | M1 | 3.1a |
| | $=\frac{9\left(\frac{2}{x-2}-1\right)^2}{4\left(\frac{2}{x-2}-1\right)\left(\frac{2}{x-2}+1\right)}=\frac{9\left(\frac{2}{x-2}-1\right)}{4\left(\frac{2}{x-2}+1\right)}=\frac{9(4-x)}{4x}$ * | A1* | 1.1b |
| | Alternative for M1A1: $y^2=\frac{9(\cosh\theta-1)^2}{4\sinh^2\theta}=\frac{9(\cosh\theta-1)^2}{4(\cosh\theta-1)(\cosh\theta+1)}=\frac{9(\cosh\theta-1)}{4(\cosh\theta+1)}$ $\frac{9(4-x)}{4x}=\frac{9\left(4-2-\frac{2}{\cosh\theta}\right)}{8+\frac{8}{\cosh\theta}}=\frac{9(\cosh\theta-1)}{4(\cosh\theta+1)}\Rightarrow y^2=\frac{9(4-x)}{4x}$ | | |
| | $p=2$ **or** $q=4$ | M1 | 3.1a |
| | $p=2$ **and** $q=4$ | A1 | 1.1b |
| | | **(11)** | |

| | | | |
|---|---|---|---|
| **(b)** | $b^2 = a^2\left(e^2 - 1\right) \Rightarrow 9 = 16\left(e^2 - 1\right) \Rightarrow e = \frac{5}{4}$<br>Focus is at $x = ae = 4 \times \frac{5}{4} = 5$ | M1 | 1.1b |
| | $d >$ "5" $- 4 = \ldots$ | M1 | 3.1a |
| | $d > 1$* | A1* | 1.1b |
| | | **(3)** | |
| | | | **(14 marks)** |

| **Notes** |
|---|
| (a)<br>M1: Attempts to solve the problem by using differentiation to obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\theta$. Allow this mark for $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \alpha x - \beta yy' = 0 \Rightarrow y' = \ldots$ or an attempt to differentiate $x$ and $y$ wrt $\theta$ and then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta} = \ldots$<br>M1: Correct straight line method using the coordinates of $P$ and their gradient in terms of $\theta$<br>**Allow the results for the first 2 M marks to be "quoted", but any statements must be correct to score the marks.**<br>A1: Uses $y = 0$ to deduce the correct coordinates (or value of $x$) for the point $A$. Allow in any form, simplified or unsimplified (e.g. unsimplified: $x = 4\cosh\theta - 4\tanh\theta\sinh\theta$)<br>B1: Deduces that the equation of $l_2$ is $x = 4$ (may be implied by $x = 4$ used to find $y$ coordinate of $B$)<br>M1: Realises that $x = 4$ is all that is needed for the second line and substitutes this into the first line in order to find the point $B$<br>A1: Deduces the correct coordinates or $y$ value for $B$<br>(e.g. unsimplified $y = \frac{3}{\tanh\theta} - \frac{3\cosh\theta}{\tanh\theta} + 3\sinh\theta$)<br>M1: Uses a correct method for the midpoint of $AB$ (coordinates must be the right way round). This may be seen as the coordinates written separately e.g. $x = \ldots, y = \ldots$<br>M1: Having found the midpoint, identifies a correct strategy that will enable a Cartesian equation to be found. E.g. find $\cosh\theta$ in terms of $x$ and substitutes into $y$ or $y^2$ to obtain an equation in terms of $y$ and $x$ only. Mark positively here, so allow the mark if the candidate makes progress in eliminating $\theta$ even if there are slips in the working.<br>A1*: Obtains the printed answer with no errors<br>**Alternative for the previous 2 marks:** Substitutes the coordinates of their midpoint into both sides of the given equation in an attempt to show they are equal. Again mark positively but having made the substitution, some progress needs to be made in showing that both sides are equal. **For this method there must be a minimal conclusion for the A1 e.g. tick, hence true etc.**<br>**Note that these 2 marks can also be attempted by expressing the midpoint in terms of** exponentials – if you are in doubt whether to award marks seek advice from your Team Leader.<br>M1: For $p = 2$ **or** $q = 4$<br>A1: For $p = 2$ **and** $q = 4$ |

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|---|
| (b)<br>M1: A complete method for finding the $x$ coordinate of the focus using a correct eccentricity formula to find a value for e and then calculating 4e<br>M1: Completes the problem by subtracting 4 from the $x$ coordinate of the focus<br>A1*: Correct answer<br>**If you come across correct attempts using Pythagoras to prove the result send to review.** |