Series (CP2)

Questions

Q1.

$$y = \ln\left(\frac{1}{1-2x}\right), \quad |x| < \frac{1}{2}$$

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$

(4)

(b) Hence, or otherwise, find the series expansion of $\ln\left(\frac{1}{1-2x}\right)$ about $x = 0$, in ascending powers of x, up to and including the term in x^3. Give each coefficient in its simplest form.

(3)

(c) Use your expansion to find an approximate value for $\ln\left(\frac{3}{2}\right)$, giving your answer to 3 decimal places.

(3)

(Total for question = 10 marks)

Q2.

(a) Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} (3r^2 - 17r - 25) = n(n^2 - An - B)$$

where A and B are integers to be determined.

(4)

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{3k} r\tan(60r)^\circ = -k\sqrt{3}$$

(2)

Using the results from part (a) and part (b) and showing all your working,

(c) determine any value of n that satisfies

$$\sum_{r=5}^{n}\left(3r^2-17r-25\right)=15\left[\sum_{r=6}^{3n} r\tan\left(60r\right)^\circ\right]^2$$

(6)

(Total for question = 12 marks)

Q3.

f (x) = arcsin x $\qquad -1 \leq x \leq 1$

(a) Determine the first two non-zero terms, in ascending powers of x, of the Maclaurin series for f(x), giving each coefficient in its simplest form.

(4)

(b) Substitute $x=\frac{1}{2}$ into the answer to part (a) and hence find an approximate value for π

Give your answer in the form $\frac{p}{q}$ where p and q are integers to be determined.

(2)

(Total for question = 6 marks)

Q4.

Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n}\frac{1}{(r+1)(r+2)(r+3)}=\frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(Total for question = 5 marks)

Q5.

$$y = \sin x \sinh x$$

(a) Show that $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

(Total for question = 10 marks)

Q6.

(a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a \ln\left(\dfrac{b}{c}\right)$ where a, b and c are integers to be determined.

(3)

(Total for question = 7 marks)

Q7.

$$y = \cosh^n x \qquad n \geq 5$$

(a) (i) Show that

$$\frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1)\cosh^{n-2} x$$

(4)

(ii) Determine an expression for $\frac{d^4y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form.

(2)

(Total for question = 8 marks)

Q8.

Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

(Total for question = 5 marks)

Q9.

(a) Show that, for $r > 0$

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$$

(1)

(b) Hence prove that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

(3)

(c) Show that, for $n \in \mathbb{N}$, $n > 1$

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = \frac{an^2 + bn + c}{n^2(3n+1)^2}$$

where a, b and c are constants to be found.

(3)

(Total for question = 7 marks)

Q10.

(a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n} (3r-2)^2 = \frac{1}{2}n\left[6n^2 - 3n - 1\right]$$

for all positive integers n.

(5)

(b) Hence find any values of n for which

$$\sum_{r=5}^{n} (3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$$

(5)

(Total for question = 10 marks)

Q11.

(a) Show that, for $r > 0$

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}$$

(2)

(b) Hence prove, using the method of differences, that

$$\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \frac{n(n^2 + an + b)}{2(n+2)}$$

where a and b are constants to be found.

(5)

(Total for question = 7 marks)

<u>Mark Scheme</u> – Series (CP2)

Q1.

Question Number	Scheme	Notes	Marks
	$y=\ln\left(\frac{1}{1-2x}\right)$		
(a)	$y=\ln(1-2x)^{-1}=(\ln 1)-\ln(1-2x)$ $\frac{dy}{dx}=-\frac{1}{1-2x}\times -2\left(=\frac{2}{1-2x}\right)$	M1: $\frac{dy}{dx}=\frac{-1}{(1-2x)}\times\frac{d(1-2x)}{dx}$ Must use chain rule ie $\frac{k}{1-2x}$ with $k\neq\pm1$ needed. Minus sign may be missing. A1: Correct derivative	M1A1
OR	$\frac{dy}{dx}=(1-2x)\times-(1-2x)^{-2}\times-2$ $\left(=\frac{2}{1-2x}\right)$	M1: $\frac{dy}{dx}=\frac{1}{(1-2x)^{-1}}\times\frac{d(1-2x)^{-1}}{dx}$ Must use chain rule. Minus sign may be missing. A1: Correct derivative	M1A1
	$\frac{d^2y}{dx^2}=-2\times(1-2x)^{-2}\times-2$ $\left(=\frac{4}{(1-2x)^2}\right)$	Correct second derivative obtained from a correct first derivative.	A1
	$\frac{d^3y}{dx^3}=-8\times(1-2x)^{-3}\times-2$ $\left(=\frac{16}{(1-2x)^3}\right)$	Correct third derivative obtained from correct first and second derivatives	A1
			(4)
	Alternative by use of exponentials and implicit differentiation		
(a)	$y=\ln\left(\frac{1}{1-2x}\right)\Rightarrow e^y=\frac{1}{1-2x}=(1-2x)^{-1}$		
	$e^y\frac{dy}{dx}=2(1-2x)^{-2}$	Differentiates using implicit differentiation and chain rule.	M1
	$\frac{dy}{dx}=2e^{-y}(1-2x)^{-2}$ or $\frac{2}{(1-2x)}$	Correct derivative in either form. Equivalents accepted.	A1
	If $\frac{dy}{dx}=\frac{2}{(1-2x)}$ has been used from here, see main scheme for second and third derivatives		

(b)	$(y_0 = 0), y'_0 = 2, y''_0 = 4, y'''_0 = 16$	Attempt values at $x = 0$ using their derivatives from (a) $y_0 = 0$ need not be seen but other 3 values must be attempted.	M1
	$(y =)(0) + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!}$	Uses their values in the correct Maclaurin series. Must see x^3 term Can be implied by a final series which is correct for their values. 2!,3! or 2 and 6	M1
	$y = 2x + 2x^2 + \frac{8}{3}x^3$	Correct expression. Must start $y = ...$ or $\ln\left(\frac{1}{1-2x}\right) = ...$ $f(x) = ...$ allowed **only** if $f(x)$ is defined to be one of these.	A1cao
			(3)
	Alternative (b)		
	$y = \ln\left(\frac{1}{1-2x}\right) = -\ln(1-2x)$	Log power law applied correctly	M1
	$= -\left((-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3}\right)$	Replaces x with $-2x$ in the expansion for $\ln(1+x)$ (in formula book)	M1
	$y = 2x + 2x^2 + \frac{8}{3}x^3$	Correct expression	A1cao
(c)	$\frac{1}{1-2x} = \frac{3}{2} \Rightarrow x = \frac{1}{6}$	Correct value for x, seen explicitly or substituted in their expansion	B1
	$\ln\left(\frac{3}{2}\right) \approx 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right)^2 + \frac{8}{3}\left(\frac{1}{6}\right)^3$	Substitute their value of x into their expansion. May need to check this is correct for their expansion and their x. (Calculator value for $\ln\left(\frac{3}{2}\right)$ is 0.405)	M1
	$= 0.401$	Must come from correct work	A1cso
NB:	$\ln 3 - \ln 2$ or $\ln 3 + \ln\left(\frac{1}{2}\right)$ scores 0/3 as $\lvert x\rvert$ must be $< \frac{1}{2}$		
	Answer with no working scores 0/3		**(3)**
			Total 10

Q2.

Question	Scheme	Marks	AOs
(a)	$\sum_{r=1}^{n}\left(3r^2-17r-25\right)=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-\ldots$	M1	1.1b
	$=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-25n$	A1	1.1b
	$=n\left(\frac{1}{2}\left(2n^2+3n+1\right)-\frac{17}{2}(n+1)-25\right)$ or $=\frac{n}{2}\left(\left(2n^2+3n+1\right)-17(n+1)-50\right)$	M1	1.1b
	$=n\left(n^2-7n-33\right)$ cso (so $A=7$ and $B=33$)	A1 cso	2.1
		(4)	
(b)	$\sum_{r=1}^{3k} r\tan(60r)°$ $=\tan(60)°+2\tan(120)°+3\tan(180)°+4\tan(240)°+5\tan(300)°+6\tan(360)°+$ $=\left(\sqrt{3}-2\sqrt{3}+0\right)+\left(4\sqrt{3}-5\sqrt{3}+0\right)+\ldots$	M1	3.1a
	Since tan has period 180° we see $\tan(60r)°$ repeats every three terms and each group of three terms results in $-\sqrt{3}$ as a sum, so with k groups of terms the sum is $-k\sqrt{3}$	A1	2.4
		(2)	

(c)	$\sum_{r=5}^{n}\left(3r^2-17r-25\right)=\sum_{r=1}^{n}\left(3r^2-17r-25\right)-\sum_{r=1}^{4}\left(3r^2-17r-25\right)$	M1	1.1b
	$=n\left(n^2-7n-33\right)-4\left(4^2-7\times4-33\right)$ $\left(=n\left(n^2-7n-33\right)+180\right)$	A1	1.1b
	$\sum_{r=6}^{3n} r\tan(60r)°=-n\sqrt{3}+2\sqrt{3}$ (allow for $-n\sqrt{3}--2\sqrt{3}$)	B1	2.2a
	$\Rightarrow n\left(n^2-7n-33\right)+180=15\left[-n\sqrt{3}+2\sqrt{3}\right]^2$ $\Rightarrow n^3-7n^2-33n+180=15\left(3n^2-12n+12\right)$ $\Rightarrow n^3-52n^2+147n=0$	M1	3.1a
	$\Rightarrow n^3-52n^2+147n=0\Rightarrow n=\ldots$	M1	1.1b
	But need $n>5$ for sums to be valid, so $n=49$ (allow if $n=0$ also given but $n=3$ must be rejected).	A1	2.3
		(6)	
			(12 marks)

Notes:
(a) **M1**: Applies the formulas for sum of integers and sum of squares of integers to the summation. **A1**: Correct unsimplified expression for the sum, including the $25n$ **M1**: Expands and factors out the n or ½n **A1**: Correct proof, no errors seen. **(b)** **M1**: Writes out first few terms of the sum, at least 3, and identifies the repeating pattern, e.g. through bracketed terms or stating sum repeat every three terms oe. **A1**: Correct explanation identifying $-\sqrt{3}$ is the sum of each group of three terms, so with k lots of three terms the sum is $-k\sqrt{3}$ **(c)** **M1**: Applies formula from (a) to left-hand side as a difference of two summations with either 4 or 5 as the limit on the second sum. **A1**: Correct expression for the left-hand side in terms of n **B1**: Correct expression for the sum on the right-hand side, allow if it arises from lower limit 6 used instead of 5 as the 6^{th} term is zero. May subtract the first few terms directly from the work in (b). **M1**: Both sides expanded and terms gathered to reach a simplified cubic equation for n with no other unknowns (may not have factor of n if errors made, which is fine for the method mark). This mark is not dependent on any previous marks and can be awarded as long as there is an attempt at both sides of the equation and an attempt at squaring their $\sum_{r=6}^{3n} r\tan(60r)°$. If divides through by n this mark is awarded for a 3TQ **M1**: Solves their cubic equation, which may be via calculator (so may need to check values). They may divide by n and solve a quadratic. Condone decimal roots truncated or rounded **A1**: Selects the correct value of n to give 49 as the only non-trivial answer. The value 3 must be rejected as summation on left undefined for this value, but accept if 0 and 49 are given (since both sides evaluate to 0 for $n = 0$ depending on one's interpretation of summations).

Q3.

Question	Scheme	Marks	AOs
(a)	$f'(x) = A(1-x^2)^{-\frac{1}{2}}$ $\quad f''(x) = Bx(1-x^2)^{-\frac{3}{2}}$ and $f'''(x) = C(1-x^2)^{-\frac{3}{2}} + Dx^2(1-x^2)^{-\frac{5}{2}}$ or $\dfrac{C(1-x^2)^{\frac{3}{2}} + Dx^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$	M1	2.1
	$f'(x) = (1-x^2)^{-\frac{1}{2}}$ or $\dfrac{1}{\sqrt{1-x^2}}$ $\quad f''(x) = x(1-x^2)^{-\frac{3}{2}}$ or $\dfrac{x}{(1-x^2)^{\frac{3}{2}}}$ and $f'''(x) = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$ or $\dfrac{1}{(1-x^2)^{\frac{3}{2}}} + \dfrac{3x^2}{(1-x^2)^{\frac{5}{2}}}$ from quotient rule $\dfrac{(1-x^2)^{\frac{3}{2}} + 3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$	A1	1.1b
	Finds $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and applies the formula $f(x) = f(0) + f'(0)x + f''(0)\dfrac{x^2}{2} + f'''(0)\dfrac{x^3}{6}$ $\{f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = 1\}$	M1	1.1b
	$f(x) = x + \dfrac{x^3}{6}$ cso	A1	1.1b
		(4)	
(b)	$\arcsin\left(\dfrac{1}{2}\right) = \dfrac{1}{2} + \dfrac{\left(\frac{1}{2}\right)^3}{6} = \dfrac{\pi}{6} \Rightarrow \pi = \ldots$	M1	1.1b
	$\pi = \dfrac{25}{8}$ o.e.	A1ft	2.2b
		(2)	
			(6 marks)

Notes:
(a) **M1:** Finds the correct form of the first three derivatives, may be unsimplified – the third may come later. **A1:** Correct first three derivatives, may be unsimplified – the third may come later. **M1:** Finds $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and applies to the correct formula, needs to go up to x^3. **A1:** $x + \dfrac{x^3}{6}$ cso ignore any higher terms whether correct or not Special case: If they think that their $f''(0) \neq 0$ then maximum score M1 A0 M1 A0 M1 for correct form of the first two derivatives M1 Correctly uses their $f(0)$, $f'(0)$, $f''(0)$ and applies to the correct formula

Note: If candidates do not find the first three derivatives but use $f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=1$ and use these correctly in the formula this can score M0 A0 M1 A0
(b) **M1:** Substitutes $x=\frac{1}{2}$ into both sides and rearranges to find $\pi = \ldots$ **A1ft:** Infers that $\pi=\frac{25}{8}$ o.e. Follow through their $6\mathrm{f}\left(\frac{1}{2}\right)$

Q4.

Question	Scheme	Marks	AOs
	$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3} \Rightarrow A=\ldots,\ B=\ldots,\ C=\ldots$ $\left(\text{NB } A=\frac{1}{2}\ \ B=-1\ \ C=\frac{1}{2}\right)$	M1	3.1a
	$r=0$ $\quad \frac{1}{2}\left[\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right]$ or $\frac{1}{2\;1}-\frac{1}{2}+\frac{1}{2\;3}$ or $\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$		
	$r=1$ $\quad \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$ or $\frac{1}{2\;2}-\frac{1}{3}+\frac{1}{2\;4}$ or $\frac{1}{4}-\frac{1}{3}+\frac{1}{8}$		
	$r=n-1$ $\quad \frac{1}{2}\left[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right]$ or $\frac{1}{2\;n}-\frac{1}{n+1}+\frac{1}{2\;n+2}$ or $\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2n+4}$	M1	2.1
	$r=n$ $\quad \frac{1}{2}\left[\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}\right]$ or $\frac{1}{2\;n+1}-\frac{1}{n+2}+\frac{1}{2\;n+3}$ or $\frac{1}{2n+2}-\frac{1}{n+2}+\frac{1}{2n+6}$		
	$\frac{1}{2}-\frac{1}{2}+\frac{1}{4}+\frac{1}{2(n+2)}-\frac{1}{n+2}+\frac{1}{2(n+3)}$ or $\frac{1}{4}-\frac{1}{2(n+2)}+\frac{1}{2(n+3)}$	A1	1.1b
	$=\frac{n^2+5n+6+2n+6-4n-12+2n+4}{4(n+2)(n+3)}$	M1	1.1b
	$=\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	A1	2.2a
		(5)	
			(5 marks)

Notes
M1: A complete strategy to find A, B and C e.g. partial fractions. Allow slip when finding the constant but must be the correct form of partial fractions and correct identity. M1: Starts the process of differences to identify the relevant fractions at the start and end. Must have attempted a minimum of $r=0,\quad r=1,\quad \ldots\quad r=n-1$ and $r=n$ Follow through on their values of A, B and C. Look for $r=0\rightarrow \frac{A}{1}-\frac{B}{2}+\frac{C}{3}$ $\qquad r=1\rightarrow \frac{A}{2}-\frac{B}{3}+\frac{C}{4}$ $r=n-1\rightarrow \frac{A}{n}-\frac{B}{n+1}+\frac{C}{n+2}$ $\qquad r=n\rightarrow \frac{A}{n+1}-\frac{B}{n+2}+\frac{C}{n+3}$ A1: Correct fractions from the beginning and end that do not cancel stated. M1 Combines all 'their' fractions (at least two algebraic fractions) over their correct common denominator, does not need to be the lowest common denominator (allow a slip in the numerator). A1: Correct answer. **Note**: if they start with $r=1$ the maximum they can score is M1M0A0M1A0 **Note:** Proof by induction gains no marks

Q5.

Question	Scheme	Marks	AOs
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \cosh x + \cos x \sinh x$	M1	1.1a
	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \cos x \cosh x + \sin x \sinh x + \cos x \cosh x - \sin x \sinh x$ $(= 2\cos x \cosh x)$	M1	1.1b
	$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2\cos x \sinh x - 2\sin x \cosh x$	M1	1.1b
	$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -4\sinh x \sin x = -4y$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 = 2,\ \left(\frac{\mathrm{d}^6 y}{\mathrm{d}x^6}\right)_0 = -8,\ \left(\frac{\mathrm{d}^{10} y}{\mathrm{d}x^{10}}\right)_0 = 32$	B1	3.1a
	Uses $y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots$ with their values	M1	1.1b
	$= \frac{x^2}{2!}(2) + \frac{x^6}{6!}(-8) + \frac{x^{10}}{10!}(32)$	A1	1.1b
	$= x^2 - \frac{x^6}{90} + \frac{x^{10}}{113400}$	A1	1.1b
		(4)	
(c)	$2(-4)^{n-1}\frac{x^{4n-2}}{(4n-2)!}$	M1 A1	3.1a 2.2a
		(2)	
		(10 marks)	

Notes

(a)

M1: Realises the need to use the product rule and attempts first derivative

M1: Realises the need to use a second application of the product rule and attempts the second derivative

M1: Correct method for the third derivative

A1*: Obtains the correct 4$^{\text{th}}$ derivative and links this back to y

(b)

B1: Makes the connection with part (a) to establish the general pattern of derivatives and finds the correct non-zero values

M1: Correct attempt at Maclaurin series with their values

A1: Correct expression un-simplified

A1: Correct expression and simplified

(c)

M1: Generalising, dealing with signs, powers and factorials.

A1: Correct expression.

Q6.

Question	Scheme	Marks	AOs
(a)	Applies $ln\left(\frac{r+1}{r-1}\right) = ln(r+1) - ln(r-1)$ to the problem in order to apply differences.	M1	3.1a
	$\sum_{r=2}^{n}(ln(r+1) - ln(r-1))$ $= (ln(3) - ln(1)) + (ln(4) - ln(2)) + (ln(5) - ln(3)) + \ldots$ $+ (ln(n) - ln(n-2)) + (ln(n+1) - ln(n-1))$	dM1	1.1b
	$ln(n) + ln(n+1) - ln\,2$	A1	1.1b
	$ln\left(\frac{n(n+1)}{2}\right)$ * cso	A1 *	2.1
		(4)	
(b)	$\sum_{r=51}^{100} ln\left(\frac{r+1}{r-1}\right) = \sum_{r=2}^{100} ln\left(\frac{r+1}{r-1}\right) - \sum_{r=2}^{50} ln\left(\frac{r+1}{r-1}\right)$ $= ln\left(\frac{100 \times 101}{2}\right) - ln\left(\frac{50 \times 51}{2}\right)$	M1	1.1b
	$\Sigma_{r=51}^{100} ln\left(\frac{r+1}{r-1}\right)^{35} = 35\,ln\left(\frac{100 \times 101}{2} \div \frac{50 \times 51}{2}\right)$	M1	3.1a
	$= 35\,ln\left(\frac{202}{51}\right)$	A1	1.1b
		(3)	
		(7 marks)	

Notes:

(a)

M1: Uses the subtraction laws of logs to start the method of differences process.

dM1: Demonstrates the method of differences process, should have a minimum of e.g $r = 2$, $r = 3$, $r = 4$, $r = n - 1$ and $r = n$ shown -- enough to establish *at least one cancelling term* and *all non-disappearing terms* though the latter may be implied by correct extraction if only the first few cases are shown. Allow this mark if an extra term for $r = 1$ has been included.

A1: Correct terms that do not cancel - must not contradict their list of terms so e.g. if $r = 1$ was included, then A0A0 follows. The ln 1 may be included for this mark.

A1*: Achieves the printed answer, with no errors or omissions **and** must have had a complete list (as per dM1) before extraction (but condone missing brackets on ln terms). If working with r throughout, they must replace by n to gain the last A, but all other marks are available.

NB For attempts at combining log terms instead of using differences, full marks may be awarded for the equivalent steps, but attempts that do not make progress in combining terms will score no marks.

(b) Condone a bottom limit of 0 or 1 being used throughout part (b).

M1: Attempts to split into (the sum up to 100) – (the sum up to k) where k is 49, 50 or 51 **and** apply the result of (a) in some way. Condone slips with the power.

M1: Having attempted to apply (a), uses difference and power log laws correctly to reach an expression of the required form.

A1: Correct answer. Accept equivalents in required form, such as $35\,ln\frac{5050}{1275}$

Q7.

Question	Scheme	Marks	AOs
(a)(i)	$\frac{dy}{dx} = \ldots \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^{n-1} x \cosh x$ **Alternatively** $y = \left(\frac{e^x+e^{-x}}{2}\right)^n$ leading to $\frac{dy}{dx} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + \ldots \left(\frac{e^x+e^{-x}}{2}\right)^n$	M1	1.1b
	$\frac{dy}{dx} = n \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x \sinh^2 x + n \cosh^n x$ **Alternatively** $\frac{dy}{dx} = n\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = n(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + n\left(\frac{e^x+e^{-x}}{2}\right)^n$	A1	2.1
	$\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x (\cosh^2 x - 1) + n \cosh^n x$	M1	2.1
	$\frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x$* cso	A1*	1.1b
		(4)	

(a)(ii)	$\frac{d^3y}{dx^3} = \ldots \cosh^{n-1} x \sinh x - \ldots \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4}$ $= \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^n x - \ldots \cosh^{n-4} x \sinh^2 x - \ldots \cos$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3 \cosh^{n-1} x \sinh x - n(n-1)(n-2) \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^3(n-1) \cosh^{n-2} x \sinh^2 x + n^3 \cosh^n x$ $-n(n-1)(n-2)(n-3) \cosh^{n-4} x \sinh^2 x - n(n-1)(n$ $-2) \cosh^{n-2} x$	A1	1.1b
		(2)	
	Alternative 1 using $\frac{d^2y}{dx^2} = n^2 y - n(n-1) \cosh^{n-2} x$ leading to $\frac{d^3y}{dx^3} = n^2 \frac{dy}{dx} - \ldots \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^2 \frac{d^2y}{dx^2} - \ldots \cosh^{n-4} x \sinh^2 x - \ldots \cosh^{n-2} x$	M1	1.1b

	$\frac{d^3y}{dx^3} = n^2\frac{dy}{dx} - n(n-1)(n-2)\cosh^{n-3}x\sinh x$ $\frac{d^4y}{dx^4} = n^2\frac{d^2y}{dx^2} - n(n-1)(n-2)(n-3)\cosh^{n-4}x\sinh^2 x - n(n-1)(n-2)\cosh^{n-2}x$	A1	1.1b
		(2)	
	Alternative 2 $y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2}x$ $y = \cosh^{n-2}x \Rightarrow \frac{d^2y}{dx^2} = \ldots\cosh^{n-2}x - \ldots\cosh^{n-4}x$ $\frac{d^4y}{dx^4} = n^2[n^2\cosh^n x - n(n-1)\cosh^{n-2}x] - n(n-1)[\ldots\cosh^{n-2}x - \ldots\cosh^{n-4}x]$	M1	1.1b
	$y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2}x$ $y = \cosh^{n-2}x \Rightarrow \frac{d^2y}{dx^2} = (n-2)^2\cosh^{n-2}x - (n-2)(n-3)\cosh^{n-4}x$ $\frac{d^4y}{dx^4} = n^2[n^2\cosh^n x - n(n-1)\cosh^{n-2}x] - n(n-1)[(n-2)^2\cosh^{n-2}x - (n-2)(n-3)\cosh^{n-4}x]$	A1	1.1b

		(2)	
	Alternative 3 Using $\frac{d^2y}{dx^2} = n^2\left(\frac{e^x+e^{-x}}{2}\right)^n - n(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ leading to $\frac{d^3y}{dx^3} = \ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right) - \ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-3}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^4y}{dx^4} = \ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + \ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ $-\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-4}\left(\frac{e^x-e^{-x}}{2}\right)^2 - \ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right) - n(n-1)(n-2)\left(\frac{e^x+e^{-x}}{2}\right)^{n-3}\left(\frac{e^x-e^{-x}}{2}\right)$	A1	1.1b

	$\frac{d^4y}{dx^4} = n^3(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + n^3\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ $-n(n-1)(n-2)(n-3)\left(\frac{e^x+e^{-x}}{2}\right)^{n-4}\left(\frac{e^x-e^{-x}}{2}\right)^2 - n(n$ $-1)(n-2)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$		
		(2)	
(b)	When $x = 0$ $y = 1,\ \ y' = 0\ ,\ \ y'' = n^2 - n(n-1),\ \ y^{(3)} = 0,$ $y^{(4)} = n^3 - n(n-1)(n-2)$ Uses their values in the expansion$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0)+\ldots$	M1	1.1b
	$y = 1 + \frac{nx^2}{2} + \frac{(3n^2-2n)x^4}{24} + \ldots$ cso	A1	2.5
		(2)	
			(8 marks)

Notes:
(a)(i)   **M1:** Uses the chain rule and product rule to find the first and second derivatives which must be of the required form, condone sign slips   Alternatively uses the exponential definition and uses the chain rule and product rule to find the first and second derivatives which must be of the required form.   **A1:** Correct unsimplified first and second derivatives, may be in exponential form.   **M1:** Uses the identity $\pm cosh^2 x \pm sinh^2 x = 1$ **A1*:** Achieves the printed answer with no errors or omissions e.g. missing x's
(a)(ii)   **M1:** Uses the chain rule and product rule to find the third and fourth derivatives which must be of the required form, condone sign slips   **A1:** Correct fourth derivative, does not need to be simplified ISW   **Alternative 1**   **M1:** Using $\frac{d^2y}{dx^2} = n^2y - n(n-1)\,cosh^{n-2} x$ to find the third and fourth derivatives which must be of the required form, condone sign slips   **A1:** Correct fourth derivative, does not need to be simplified ISW   **Alternative 2**   **M1:** Using$y = cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2\,cosh^n x - n(n-1)\,cosh^{n-2} x$ $y = cosh^{n-2} x \Rightarrow \frac{d^2y}{dx^2} = \ldots cosh^{n-2} x - \ldots cosh^{n-4} x$ leading to

$\frac{d^4y}{dx^4} = n^2[n^2 \cosh^n x - n(n-1)\cosh^{n-2} x] - n(n-1)\left[\text{their}\frac{d(\cosh^{n-2} x)}{dx}\right]$ **A1:** Correct fourth derivative, does not need to be simplified ISW **Alternative 3** **M1:** Uses the exponential definition and uses the chain rule and product rule to find the third and fourth derivatives which must be of the required form. **A1:** Correct fourth derivative, does not need to be simplified ISW
(b) **M1:** Attempts the evaluation of all four of their derivatives at $x = 0$ and applies the Maclaurin formula with their values. Note that $y^{(1)}(0) = 0$ and $y^{(3)}(0) = 0$ may be implied as they will have a multiple of sinh0. If their $y^{(3)}(0) \neq 0$ they allow this mark for their first 3 non-zero terms **A1:** Correct simplified expansion from correct derivatives cso

Q8.

Question	Scheme	Marks	AOs
	$\frac{1}{(r+1)(r+3)} \equiv \frac{A}{(r+1)} + \frac{B}{(r+3)} \Rightarrow A = ..., B = ...$	M1	3.1a
	$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} =$ $\frac{1}{2\times2} - \frac{1}{2\times4} + \frac{1}{2\times3} - \frac{1}{2\times5} + ... + \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$	M1	2.1
	$= \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$	A1	2.2a
	$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$	M1	1.1b
	$= \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	

	Alternative by Induction:		
	$n=1 \Rightarrow \frac{1}{8}=\frac{a+b}{12\times3\times4},\ n=2 \Rightarrow \frac{1}{8}+\frac{1}{15}=\frac{2(2a+b)}{12\times4\times5}$ $a+b=18,\ 2a+b=23 \Rightarrow a=...,\ b=...$	M1	3.1a
	Assume true for $n=k$ so $\sum_{r=1}^{k}\frac{1}{(r+1)(r+3)}=\frac{k(5k+13)}{12(k+2)(k+3)}$		
	$\sum_{r=1}^{k+1}\frac{1}{(r+1)(r+3)}=\frac{k(5k+13)}{12(k+2)(k+3)}+\frac{1}{(k+2)(k+4)}$	M1	2.1
	$\frac{k(5k+13)}{12(k+2)(k+3)}+\frac{1}{(k+2)(k+4)}=\frac{k(5k+13)(k+4)+12(k+3)}{12(k+2)(k+3)(k+4)}$	A1	2.2a
	$=\frac{5k^3+33k^2+52k+12k+36}{12(k+2)(k+3)(k+4)}=\frac{(k+1)(k+2)(5k+18)}{12(k+2)(k+3)(k+4)}$	M1	1.1b
	$=\frac{(\underline{k+1})(5(\underline{k+1})+13)}{12(\underline{k+1}+2)(\underline{k+1}+3)}$ So true for $n=k+1$ So $\sum_{r=1}^{n}\frac{1}{(r+1)(r+3)}=\frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	
			(5 marks)

Notes:
(Main Scheme) **M1**: Valid attempt at partial fractions **M1**: Starts the process of differences to identify the relevant fractions at the start and end **A1**: Correct fractions that do not cancel **M1**: Attempt common denominator **A1**: Correct answer
(Alternative by Induction) **M1**: Uses $n=1$ and $n=2$ to identify values for a and b **M1**: Starts the induction process by adding the $(k+1)^{\text{th}}$ term to the sum of k terms **A1**: Correct single fraction **M1**: Attempt to factorise the numerator **A1**: Correct answer and conclusion

Q9.

Question Number	Scheme	Notes	Marks
(a)	$\frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{(r+1)^2-r^2}{r^2(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}$	Correct proof (minimum as shown) ($(r+1)^2$ or r^2+2r+1 Can be worked in either direction.	B1
			(1)
(b)	$\sum_{r=1}^{n}\left(\frac{1}{r^2}-\frac{1}{(r+1)^2}\right)=1-\frac{1}{4}+\frac{1}{4}-\frac{1}{9}......+\left(\frac{1}{n^2}\right)-\frac{1}{(n+1)^2}$ Terms of the series with $r=1$, $r=n$ and one of $r=2, r=n-1$ should be shown.		M1
	$1-\frac{1}{(n+1)^2}$	Extracts correct terms that do not cancel	A1
	$\frac{(n+1)^2-1}{(n+1)^2}=\frac{n(n+2)}{(n+1)^2}*$	Correct completion with no errors	A1*cso
			(3)
(c)	$\sum_{r=n}^{3n}\frac{6r+3}{r^2(r+1)^2}=3\left(\frac{3n(3n+2)}{(3n+1)^2}-\frac{(n-1)(n+1)}{n^2}\right)$	Attempts to use $\mathrm{f}(3n)-(\mathrm{f}(n-1)\text{ or }\mathrm{f}(n))$ 3 may be missing	M1
	$=3\left(\frac{3n^3(3n+2)-(3n+1)^2(n^2-1)}{n^2(3n+1)^2}\right)$	Attempt at common denominator, Denom to be $n^2(3n+1)^2$ or $(n+1)^2(3n+1)^2$ Numerator to be difference of 2 quartics. 3 may be missing	dM1
	$=\frac{24n^2+18n+3}{n^2(3n+1)^2}$	cao	A1cao
			(3)
			Total 7
	Alternative for part (c)		
	$\sum_{r=n}^{3n}\frac{6r+3}{r^2(r+1)^2}=3\left(\frac{1}{n^2}-\frac{1}{(3n+1)^2}\right)$ OR: $3\left(\frac{1}{(n+1)^2}-\frac{1}{(3n+1)^2}\right)$	Attempts the difference of 2 terms (either difference accepted) 3 may be missing	M1
	$=3\left(\frac{(3n+1)^2-n^2}{n^2(3n+1)^2}\right)$	Valid attempt at common denominator for their fractions 3 may be missing	dM1
	$=\frac{24n^2+18n+3}{n^2(3n+1)^2}$	cao	A1
	If (b) and/or (c) are worked with r instead of n do **NOT** award the final A mark for the parts affected. This applies even if r is changed to n at the end.		

	Alternative for (b) - by induction. **NB:** No marks available if result in (a) is not used.		
	Assume true for $n=k$		
	$\sum_{r=1}^{k+1}\frac{2r+1}{r^2(r+1)^2}=\frac{k(k+2)}{(k+1)^2}+\frac{1}{(k+1)^2}-\frac{1}{(k+2)^2}$	Uses $\sum_{r=1}^{k}$ together with the $(k+1)$th term as 2 fractions (see (a))	M1
	$=\frac{k^2+2k+1}{(k+1)^2}-\frac{1}{(k+2)^2}$		
	$1-\frac{1}{(k+2)^2}=\frac{k^2+4k+3}{(k+2)^2}=\frac{(k+1)(k+3)}{(k+2)^2}$	Combines the 3 fractions to obtain a single fraction. Must be correct but numerator need not be factorised.	A1
	Show true for $n=1$	This must be seen somewhere	
	Hence proved by induction	Complete proof with no errors and a concluding statement.	A1

Q10.

Question	Scheme	Marks	AOs
(a)	$(3r-2)^2 = 9r^2 - 12r + 4$	B1	1.1b
	$\sum_{r=1}^{n}(9r^2 - 12r + 4) = 9\times\frac{1}{6}n(n+1)(2n+1) - 12\times\frac{1}{2}n(n+1) + \ldots$	M1	2.1
	$= 9\times\frac{1}{6}n(n+1)(2n+1) - 12\times\frac{1}{2}n(n+1) + 4n$	A1	1.1b
	$= \frac{1}{2}n\left[3(n+1)(2n+1) - 12(n+1) + 8\right]$	dM1	1.1b
	$= \frac{1}{2}n\left[6n^2 - 3n - 1\right]$*	A1*	1.1b
		(5)	
(b)	$\sum_{r=5}^{n}(3r-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1) - \frac{1}{2}(4)\left(6(4)^2 - 3\times 4 - 1\right)$	M1	3.1a
	$\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 0 - 2 + 0 + 4 + 0 - 6 + 0 + 8 + 0 - 10 + 0 + 12 + \ldots$	M1	3.1a
	$3n^3 - \frac{3}{2}n^2 - \frac{1}{2}n - 166 + 103\times 14 = 3n^3$ $\Rightarrow 3n^2 + n - 2552 = 0$	A1	1.1b
	$\Rightarrow 3n^2 + n - 2552 = 0 \Rightarrow n = \ldots$	M1	1.1b
	$n = 29$	A1	2.3
		(5)	
			(10 marks)

Notes
(a) Do not allow <u>proof by induction</u> (but the B1 could score for $(3r-2)^2 = 9r^2 - 12r + 4$ if seen) B1: Correct expansion M1: <u>**Substitutes**</u> at least one of the standard formulae into their expanded expression A1: Fully correct expression dM1: Attempts to factorise $\frac{1}{2}n$ having used at least one standard formula correctly. Dependent on the first M mark and dependent on there being an n in all terms. A1*: Obtains the printed result with no errors seen (b) M1: Uses the result from part (a) by substituting $n = 4$ and subtracts from the result in (a) in order to find the first sum in terms of n. M1: Identifies the periodic nature of the second sum by calculating terms. This may be implied by a sum of 14. A1: Uses their sum and the given result to form the correct 3 term quadratic M1: Solves their three term quadratic to obtain at least one value for n A1: Obtains $n = 29$ only or obtains $n = 29$ and $n = -\frac{88}{3}$ and rejects the $-\frac{88}{3}$

Q11.

Question Number	Scheme	Notes	Marks
(a)	$\dfrac{r(r+1)(r+2)-3(r+1)(r+2)+r+2-(r+1)}{(r+1)(r+2)}$	Attempt common denominator with at least two correct expressions in the numerator. Denominator must be seen now or later.	M1
	$=\dfrac{r^3-7r-5}{(r+1)(r+2)}$ **	No errors seen and at least one intermediate step shown.	A1 cso
			(2)
ALTs	1) Start with RHS and use partial fractions		
	2) Start with RHS, divide and then use partial fractions on the remainder.		
	For either: M1 complete method as described;	A1cso No errors seen and at least one intermediate step shown.	
(b)	$\sum_{r=1}^{n}(r-3)=\frac{1}{2}n(n+1)-3n$ or $\sum_{r=1}^{n}(r-3)=\frac{1}{2}n\left(-2+(n-3)\right)$	Use formula for sum of the natural numbers from 1 to n and '$-3n$' or either formula for the sum of an AP. If general formula not quoted the sub must be correct. (See general rules on "Use of a formula" page 7)	M1
	$\frac{1}{2}-\frac{1}{3}$ $\frac{1}{3}-\frac{1}{4}$ $\frac{1}{n}-\frac{1}{n+1}$ $\frac{1}{n+1}-\frac{1}{n+2}$	Method of differences with at least 3 lines shown, (2 at start and 1 at end or 1 at start and 2 at end). Last Line may be missing $=\frac{1}{n+1}$ Ignore extra terms at start due to including the "$(r-3)$" term in each line (or some lines).	M1
	$=\frac{1}{2}-\frac{1}{n+2}$	Extract the 2 remaining terms. Second M mark only needed.	A1
	$\sum_{r=1}^{n}=\dfrac{n(n+1)(n+2)-6n(n+2)+n+2-2}{2(n+2)}$ Or $\sum_{r=1}^{n}=\dfrac{n(n+1)(n+2)-6n(n+2)+n}{2(n+2)}$	Attempt the correct common denominator using all their terms. dependent upon previous M (but not the first). The numerators must be changed. Denominator to be present now or later.	dM1
	$=\dfrac{n(n^2-3n-9)}{2(n+2)}$	$a=-3, b=-9$ Need not be shown explicitly.	A1cso
			(5)
			Total 7