# Vectors (FP1)

## Questions

**Q1.**

With respect to a fixed origin $O$, the points $A$, $B$ and $C$ have coordinates (3, 4, 5), (10, –1, 5) and (4, 7, –9) respectively.

The plane $\Pi$ has equation $4x - 8y + z = 2$

The line segment $AB$ meets the plane $\Pi$ at the point $P$ and the line segment $BC$ meets the plane $\Pi$ at the point $Q$.

(a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5

**(6)**

The point $D$ has coordinates ($k$, 4, –1), where $k$ is a constant.

Given that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ form three edges of a parallelepiped of volume 226

(b) find the possible values of the constant $k$.

**(4)**

**(Total for question = 10 marks)**

**Q2.**

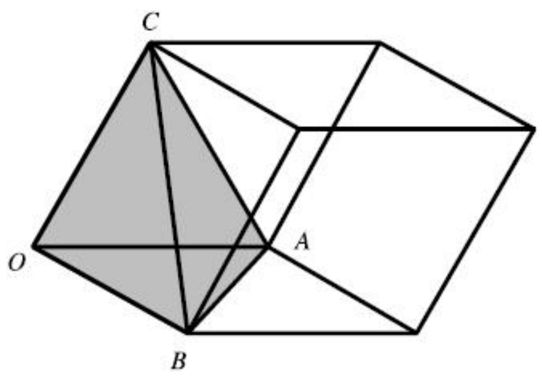


**Figure 1**

Figure 1 shows a sketch of a solid sculpture made of glass and concrete. The sculpture is modelled as a parallelepiped.

The sculpture is made up of a concrete solid in the shape of a tetrahedron, shown shaded in Figure 1, whose vertices are $O(0, 0, 0)$, $A(2, 0, 0)$, $B(0, 3, 1)$ and $C(1, 1, 2)$,where the units are in metres. The rest of the solid parallelepiped is made of glass which is glued to the concrete tetrahedron.

(a) Find the surface area of the glued face of the tetrahedron.

**(4)**

(b) Find the volume of glass contained in this parallelepiped.

**(5)**

(c) Give a reason why the volume of concrete predicted by this model may not be an accurate value for the volume of concrete that was used to make the sculpture.

**(1)**

**(Total for question = 10 marks)**

**Q3.**

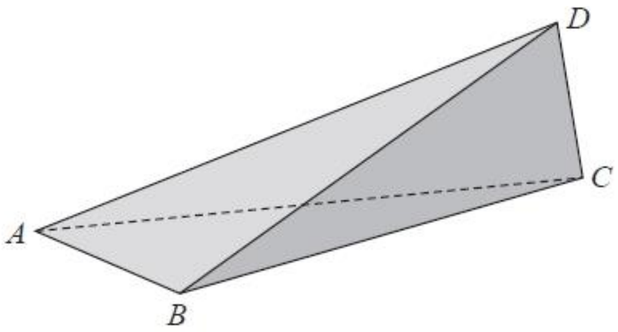


**Figure 1**

Figure 1 shows a sketch of a solid doorstop made of wood. The doorstop is modelled as a tetrahedron.

Relative to a fixed origin $O$, the vertices of the tetrahedron are $A$ (2, 1, 4), $B$ (6, 1, 2), $C$ (4, 10, 3) and $D$ (5, 8, $d$), where $d$ is a positive constant and the units are in centimetres.

(a) Find the area of the triangle $ABC$.

**(4)**

Given that the volume of the doorstop is 21 $cm^3$

(b) find the value of the constant $d$.

**(4)**

**(Total for question = 8 marks)**

**Q4.**

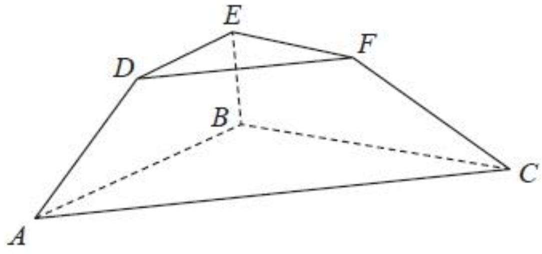


**Figure 3**

Figure 3 shows a solid display stand with parallel triangular faces *ABC* and *DEF*.
Triangle *DEF* is similar to triangle *ABC*.

With respect to a fixed origin *O*,
the points *A*, *B* and *C* have coordinates (3, –3, 1), (–5, 3, 3) and (1, 7, 5) respectively and the points *D*, *E* and *F* have coordinates (2, –1, 8), (–2, 2, 9) and (1, 4, 10) respectively.
The units are in centimetres.

(a) Show that the area of the triangular face *DEF* is $\frac{1}{2}\sqrt{339}\ \text{cm}^2$

**(3)**

(b) Find, in $\text{cm}^3$, the exact volume of the display stand.

**(7)**

**(Total for question = 10 marks)**

**Q5.**

The plane $\Pi_1$ has equation $x - 2y - 3z = 5$ and the plane $\Pi_2$ has equation $6x + y - 4z = 7$

(a) Find, to the nearest degree, the acute angle between $\Pi_1$ and $\Pi_2$

**(3)**

The point *P* has coordinates (2, 3, −1). The line *l* is perpendicular to $\Pi_1$ and passes through the point *P*. The line *l* intersects $\Pi_2$ at the point *Q*.

(b) Find the coordinates of *Q*.

**(4)**

The plane $\Pi_3$ passes through the point *Q* and is perpendicular to $\Pi_1$ and $\Pi_2$

(c) Find an equation of the plane $\Pi_3$ in the form $\mathbf{r.n} = p$

**(4)**

**(Total for question = 11 marks)**

**Q6.**

The surface of a horizontal tennis court is modelled as part of a horizontal plane, with the origin on the ground at the centre of the court, and

- **i** and **j** are unit vectors directed across the width and length of the court respectively
- **k** is a unit vector directed vertically upwards
- units are metres

After being hit, a tennis ball, modelled as a particle, moves along the path with equation

$$\mathbf{r} = (-4.1 + 9\lambda - 2.3\lambda^2)\mathbf{i} + (-10.25 + 15\lambda)\mathbf{j} + (0.84 + 0.8\lambda - \lambda^2)\mathbf{k}$$

where $\lambda$ is a scalar parameter with $\lambda \geq 0$

Assuming that the tennis ball continues on this path until it hits the ground,

(a) find the value of $\lambda$ at the point where the ball hits the ground.

**(2)**

The direction in which the tennis ball is moving at a general point on its path is given by

$$(9 - 4.6\lambda)\mathbf{i} + 15\mathbf{j} + (0.8 - 2\lambda)\mathbf{k}$$

(b) Write down the direction in which the tennis ball is moving as it hits the ground.

**(1)**

(c) Hence find the acute angle at which the tennis ball hits the ground, giving your answer in degrees to one decimal place.

**(4)**

The net of the tennis court lies in the plane $\mathbf{r}.\,\mathbf{j} = 0$

(d) Find the position of the tennis ball at the point where it is in the same plane as the net.

**(3)**

The maximum height above the court of the top of the net is 0.9 m.

Modelling the top of the net as a horizontal straight line,

(e) state whether the tennis ball will pass over the net according to the model, giving a reason for your answer.

**(1)**

With reference to the model,

(f) decide whether the tennis ball will actually pass over the net, giving a reason for your answer.

**(2)**

**(Total for question = 13 marks)**

**Q7.**

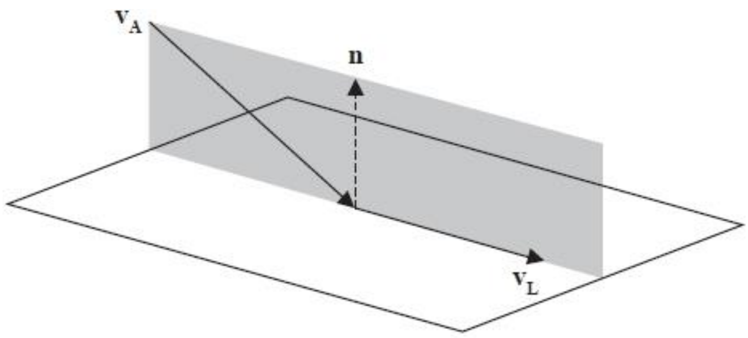


**Figure 2**

A small aircraft is landing in a field.

In a model for the landing the aircraft travels in different straight lines before and after it lands, as shown in Figure 2.

The vector $\mathbf{v}_A$ is in the direction of travel of the aircraft as it approaches the field.

The vector $\mathbf{v}_L$ is in the direction of travel of the aircraft after it lands.

With respect to a fixed origin, the field is modelled as the plane with equation

$$x - 2y + 25z = 0$$

and

$$\mathbf{v}_A = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

(a) Write down a vector **n** that is a normal vector to the field.

**(1)**

(b) Show that $\mathbf{n} \times \mathbf{v}_A = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$, where $\lambda$ is a constant to be determined.

**(2)**

When the aircraft lands it remains in contact with the field and travels in the direction $\mathbf{v}_L$

The vector $\mathbf{v}_L$ is in the same plane as both $\mathbf{v}_A$ and **n** as shown in Figure 2.

(c) Determine a vector which has the same direction as $\mathbf{v}_L$

**(3)**

(d) State a limitation of the model.

**(1)**

**(Total for question = 7 marks)**

**Q8.**

With respect to a fixed origin $O$, the points $A$, $B$ and $C$ have position vectors given by

$$\overrightarrow{OA} = 18\mathbf{i} - 14\mathbf{j} - 2\mathbf{k} \quad \overrightarrow{OB} = -7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \quad \overrightarrow{OC} = -2\mathbf{i} - 9\mathbf{j} - 6\mathbf{k}$$

The points $O$, $A$, $B$ and $C$ form the vertices of a tetrahedron.

(a) Show that the area of the triangular face $ABC$ of the tetrahedron is 108 to 3 significant figures.

**(3)**

(b) Find the volume of the tetrahedron.

**(4)**

An oak wood block is made in the shape of the tetrahedron, with centimetres taken for the units.

The density of oak is 0.85 g $cm^{-3}$

(c) Determine the mass of the block, giving your answer in kg.

**(2)**

**(Total for question = 9 marks)**

Q9.

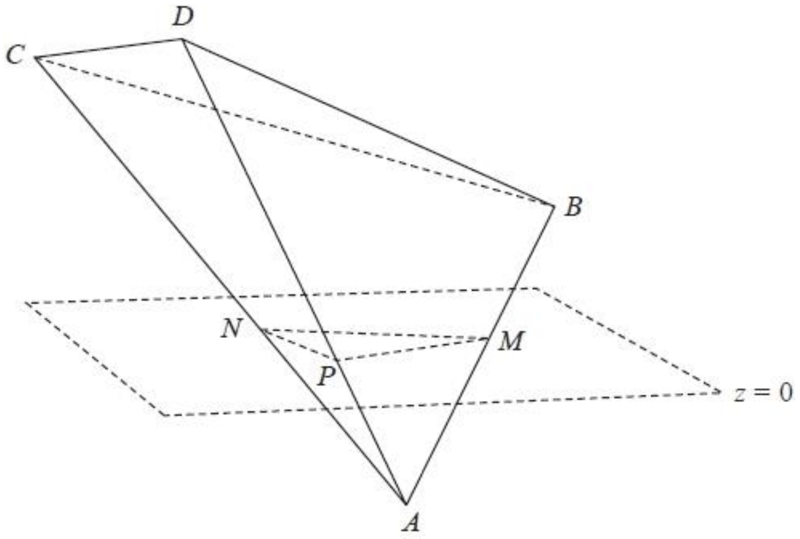


**Figure 1**

The points $A(3, 2, -4)$, $B(9, -4, 2)$, $C(-6, -10, 8)$ and $D(-4, -5, 10)$ are the vertices of a tetrahedron.

The plane with equation $z = 0$ cuts the tetrahedron into two pieces, one on each side of the plane.

The edges $AB$, $AC$ and $AD$ of the tetrahedron intersect the plane at the points $M$, $N$ and $P$ respectively, as shown in Figure 1.

Determine

(a) the coordinates of the points $M$, $N$ and $P$,

**(3)**

(b) the area of triangle $MNP$,

**(2)**

(c) the exact volume of the solid $BCDPNM$.

**(6)**

**(Total for question = 11 marks)**

**Q10.**

Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters and $a$ is a constant.

In the model, the angle between the birds' flight paths is 120°

(a) Determine the value of $a$.

**(4)**

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point.

**(5)**

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2$$

(c) Hence determine the shortest distance from the nest to the ground level of the park.

**(3)**

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer.

**(1)**

**(Total for question = 13 marks)**

**Q11.**

With respect to a fixed origin *O*, the line *l* has equation

$$(\mathbf{r} - (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k})) \times (9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = 0$$

The point *A* lies on *l* such that the direction cosines of $\overrightarrow{OA}$ with respect to the **i**, **j** and **k** axes are $\frac{3}{7}$, $\beta$ and $\gamma$.

Determine the coordinates of the point *A*.

**(7)**

**(Total for question = 7 marks)**

# Mark Scheme – Vectors (FP1)

## Q1.

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Examples:<br>Area $APQC$ = Area $ABC$ – Area $PBQ$<br>Area $APQC$ = Area $APC$ + Area $CPQ$<br>Area $APQC$ = Area $APQ$ + Area $AQC$<br>Area $APQC = \frac{1}{2}\lvert \mathbf{AQ} \times \mathbf{PC} \rvert$ | M1 | 3.1a |
| | Line $AB$: $r = \begin{pmatrix}3\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}10-3\\-1-4\\5-5\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}7\\-5\\0\end{pmatrix}$<br>**or**<br>Line $BC$: $r = \begin{pmatrix}10\\-1\\5\end{pmatrix} + \mu\begin{pmatrix}10-4\\-1-7\\5+9\end{pmatrix} = \begin{pmatrix}10\\-1\\5\end{pmatrix} + \mu\begin{pmatrix}6\\-8\\14\end{pmatrix}$ | M1 | 3.1a |
| | $4(3+7\lambda)-8(4-5\lambda)+5=2 \Rightarrow \lambda = ... \Rightarrow P$ is …<br>**or**<br>$4(10+6\mu)-8(-1-8\mu)+5+14\mu=2 \Rightarrow \mu = ... \Rightarrow Q$ is …<br>$\left(\text{NB } \lambda = \frac{1}{4},\ \mu = -\frac{1}{2}\right)$ | M1 | 2.1 |
| | $P(4.75,\ 2.75,\ 5)$ **and** $Q(7,\ 3,\ -2)$ | A1 | 1.1b |
| | Area $ABC = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 7 & -5 & 0\\ 6 & -8 & 14\end{vmatrix} = \frac{1}{2}\sqrt{70^2+98^2+26^2}$<br>Area $PBQ = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5.25 & -3.75 & 0\\ 3 & -4 & 7\end{vmatrix} = \frac{1}{2}\sqrt{26.25^2+36.75^2+9.75^2}$<br>Area $APQC = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & 1 & 7\\ 0.75 & -4.25 & 14\end{vmatrix} = \frac{1}{2}\sqrt{43.75^2+61.25^2+16.25^2}$<br>NB: Area $APQ$ = 7.7004, Area $AQC$ = 30.8018,<br>Area $CPQ$ = 23.101, Area $APC$ = 15.4008 | M1 | 2.1 |
| | Area $ABC$ – Area $PBQ$ = 38.5* | A1* | 1.1b |
| | | **(6)** | |
| (b) | $\overrightarrow{AB} = \begin{pmatrix}7\\-5\\0\end{pmatrix},\ \overrightarrow{AC} = \begin{pmatrix}1\\3\\-14\end{pmatrix},\ \overrightarrow{AD} = \begin{pmatrix}k-3\\0\\-6\end{pmatrix}$ | M1 | 3.1a |

| | | | |
|---|---|---|---|
| | $\overrightarrow{AB}\times\overrightarrow{AC}\square\overrightarrow{AD}=\begin{vmatrix} 7 & -5 & 0 \\ 1 & 3 & -14 \\ k-3 & 0 & -6 \end{vmatrix}=...$ | | |
| | $\overrightarrow{AB}\times\overrightarrow{AC}\square\overrightarrow{AD}=7\times-18+5\left(-6+14k-42\right)$ | A1 | 1.1b |
| | $7\times-18+5\left(-6+14k-42\right)=\pm 226\Rightarrow k=...$ | **dM1** | 3.1a |
| | $k=2$ or $\frac{296}{35}$ | A1 | 1.1b |
| | | **(4)** | |
| | | | **(10 marks)** |

**Notes**

(a)

M1: Identifies a correct strategy to determine the area of the required quadrilateral. The attempt does **<u>not</u> need to be complete** for this mark so one of the statements (or intentions) in the markscheme would be sufficient.

M1: Correct attempt to find the equation of the line $AB$ **or** the line $BC$

M1: Uses **at least one** of their lines and the equation of the given plane to determine the value of **at least one** of the parameters and hence the coordinates of $P$ **or** $Q$

A1: **Both** coordinates correct – allow as vectors and may be implied if for example the candidate calculates the vectors e.g. AP, AQ, CP, CQ without stating the coordinates explicitly

M1: Uses all the required information to **<u>calculate appropriate areas correctly</u>** leading to the area of the quadrilateral. Needs to be a complete method here.

A1*: Reaches 38.5 with no errors

(b)

M1: Adopts a correct strategy by finding suitable vectors and forming the scalar triple product. This is often done in 2 steps e.g.

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}70\\98\\26\end{pmatrix} \text{ or } \overrightarrow{AB}\times\overrightarrow{AD}=\begin{pmatrix}30\\42\\5k-15\end{pmatrix} \text{ or } \overrightarrow{AC}\times\overrightarrow{AD}=\begin{pmatrix}-18\\48-14k\\-3k+9\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{AD}=\begin{pmatrix}70\\98\\26\end{pmatrix}.\begin{pmatrix}k-3\\0\\-6\end{pmatrix}=70k-210-156$$

$$\text{or } \overrightarrow{AB}\times\overrightarrow{AD}.\overrightarrow{AC}=\begin{pmatrix}30\\42\\5k-15\end{pmatrix}.\begin{pmatrix}1\\3\\-14\end{pmatrix}=30+126-70k+210$$

$$\text{or } \overrightarrow{AC}\times\overrightarrow{AD}.\overrightarrow{AB}=\begin{pmatrix}-18\\48-14k\\-3k+9\end{pmatrix}.\begin{pmatrix}7\\-5\\0\end{pmatrix}=-126+70k-240$$

**If it is not clear that the vector product is being used, at least 2 of the components should be correct.**

A1: Correct expression for the triple product in terms of $k$ (should be $\pm\left(70k-366\right)$)

Ignore the presence or absence of "1/6" for the first 2 marks

**d**M1: Realises that ±226 is possible for the value of the triple product and attempts to solve to obtain 2 values for $k$. **Dependent on the previous method mark.**

A1: Correct values (must be exact)

**Q2.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Identifies glued face is triangle $ABC$ and attempts to find the area, e.g. evidences by use of $\frac{1}{2}\lvert\mathbf{AB}\times\mathbf{AC}\rvert$ | M1 | 3.1a |
| | $\frac{1}{2}\lvert\mathbf{AB}\times\mathbf{AC}\rvert = \frac{1}{2}\lvert(-2\mathbf{i}+3\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}+2\mathbf{k})\rvert$ | M1 | 1.1b |
| | $=\frac{1}{2}\lvert 5\mathbf{i}+3\mathbf{j}+\mathbf{k}\rvert$ | M1 | 1.1b |
| | $=\frac{1}{2}\sqrt{35}\,(\mathrm{m}^2)$ | A1 | 1.1b |
| | | (4) | |
| (a) ALT 1 | Identifies glued face is triangle $ABC$ and attempts to find the area, e.g. evidences by use of $\frac{1}{2}\sqrt{\lvert\mathbf{AB}\rvert^2\lvert\mathbf{AC}\rvert^2-(\mathbf{AB}.\mathbf{AC})^2}$ | M1 | 3.1a |
| | $\lvert\mathbf{AB}\rvert^2=4+9+1=14,\ \lvert\mathbf{AC}\rvert^2=1+1+4=6$ and $\mathbf{AB}.\mathbf{AC}=2+3+2=7$ | M1 | 1.1b |
| | So area of glue is $=\frac{1}{2}\sqrt{('14')('6')-('7')^2}$ | M1 | 1.1b |
| | $=\frac{1}{2}\sqrt{35}\ (\mathrm{m}^2)$ | A1 | 1.1b |
| | | (4) | |
| (b) | Volume of parallelepiped taken up by concrete is e.g. $\frac{1}{6}(\mathbf{OC}.(\mathbf{OA}\times\mathbf{OB}))$ | M1 | 3.1a |
| | $=\frac{1}{6}(\mathbf{i}+\mathbf{j}+2\mathbf{k}).(2\mathbf{i}\times(3\mathbf{j}+\mathbf{k}))$ | M1 | 1.1b |
| | $=\frac{10}{6}=\frac{5}{3}$ | A1 | 1.1b |
| | Volume of parallelepiped is 6 × volume of tetrahedron (= 10), so volume of glass is difference between these, viz. $10-\frac{5}{3}=\ldots$ | M1 | 3.1a |
| | Volume of glass $=\frac{25}{3}(\mathrm{m}^3)$ | A1 | 1.1b |
| | | (5) | |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **(b) ALT** | $-\mathbf{j}+3\mathbf{k}$ is perpendicular to both $\mathbf{OA}=2\mathbf{i}$ and $\mathbf{OB}=3\mathbf{j}+\mathbf{k}$ | M1 | 3.1a |
| | Area $AOB=\frac{1}{2}\times\lvert\mathbf{OA}\rvert\times\lvert\mathbf{OB}\rvert=\frac{1}{2}\times 2\times\sqrt{10}=\sqrt{10}$ | A1 | 1.1b |
| | $\mathbf{i}+\mathbf{j}+2\mathbf{k}-p(-\mathbf{j}+3\mathbf{k})=\mu(2\mathbf{i})+\lambda(3\mathbf{j}+\mathbf{k})\Rightarrow p=\frac{1}{2}$ <br> and so height of tetrahedron is $h=\frac{1}{2}\lvert-\mathbf{j}+3\mathbf{k}\rvert=\frac{1}{2}\sqrt{10}$ | M1 | 3.1a |
| | Volume of glass is $V=5\times$ Volume of tetrahedron <br> $=5\times\frac{1}{3}\sqrt{10}\times\frac{1}{2}\sqrt{10}$ | M1 | 1.1b |
| | $=\frac{25}{3}\left(\mathrm{m}^3\right)$ | A1 | 1.1b |
| | | **(5)** | |
| **(c)** | The glued surfaces may distort the shapes / reduce the volume of concrete. <br> Measurements in m may not be accurate. <br> The surface of the concrete tetrahedron may not be smooth. <br> Pockets of air may form when the concrete is being poured. | B1 | 3.2b |
| | | **(1)** | |
| | | **(10 marks)** | |

| **Notes** | |
|---|---|
| (a) | Accept use of column vectors throughout. |
| M1 | Shows an understanding of what is required via an attempt at finding the area of triangle $ABC$. Any correct method for the triangle area is fine. |
| M1 | Finds $\mathbf{AB}$ and $\mathbf{AC}$ or any other appropriate pair of vectors to use in the vector product and attempts to use them. |
| M1 | Correct procedure for the vector product with at least 1 correct term. |
| A1 | $\frac{1}{2}\sqrt{35}$ or exact equivalent. |
| (a) ALT | |
| M1 | As main method. |
| M1 | Finds two appropriate sides and attempts the scalar product and magnitudes of two of the sides. May use different sides to those shown. |
| M1 | Correct full method to find the area of the triangle using their two sides. |
| A1 | $\frac{1}{2}\sqrt{35}$ or exact equivalent. |
| (b) | |
| M1 | Attempts volume of concrete by finding volume of tetrahedron with appropriate method. |
| M1 | Uses the formula with correct set of vectors substituted (may not be the ones shown) and vector product attempted. |
| A1 | Correct value for the volume of concrete. |
| M1 | Attempt to find total volume of glass by multiplying their volume of concrete by 6 and subtracting their volume of concrete. May restart to find the volume of parallelepiped. |
| A1 | $\frac{25}{3}$ only, ignore reference to units. |

| Notes | |
|---|---|
| (b)<br>ALT | |
| M1 | Notes (or works out using scalar products) that $-\mathbf{j}+3\mathbf{k}$ is a vector perpendicular to both $\mathbf{OA}=2\mathbf{i}$ and $\mathbf{OB}=3\mathbf{j}+\mathbf{k}$ |
| A1 | Finds (using that **OA** and **OB** are perpendicular), area of $AOB=\sqrt{10}$. |
| M1 | Solves $\mathbf{i}+\mathbf{j}+2\mathbf{k}-p(-\mathbf{j}+3\mathbf{k})=\mu(2\mathbf{i})+\lambda(3\mathbf{j}+\mathbf{k})$ to get height of tetrahedron.<br>$\left[(\mu=\lambda=)\ p=\frac{1}{2},\ \text{so } h=\frac{1}{2}\lvert-\mathbf{j}+3\mathbf{k}\rvert=\frac{1}{2}\sqrt{10}\right]$ |
| M1 | Identifies the correct area as 5 times the volume of the tetrahedron (may be done as in main scheme via the difference). |
| A1 | $\frac{25}{3}$ only, ignore reference to units. |
| (c)<br>B1 | Any acceptable reason in context. |

**Q3.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $A(2,1,4),\ B(6,1,2),\ C(4,10,3),\ D(5,8,d)$ | | |
| (a) Way 1 | Uses appropriate vectors in a correct method to make a complete attempt to find the area of triangle $ABC$. | M1 | 3.1b |
| | $\overrightarrow{AB}=\begin{pmatrix}4\\0\\-2\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}2\\9\\-1\end{pmatrix},\ \left\{\overrightarrow{BC}=\begin{pmatrix}-2\\9\\1\end{pmatrix}\right\}$ | M1 | 1.1b |
| | e.g. $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&0&-2\\2&9&-1\end{vmatrix}=\ldots$ or $\begin{pmatrix}4\\0\\-2\end{pmatrix}\times\begin{pmatrix}2\\9\\-1\end{pmatrix}=\ldots$ | M1 | 1.1b |
| | $=18\mathbf{i}+0\mathbf{j}+36\mathbf{k}$ | | |
| | Area $ABC=\frac{1}{2}\sqrt{(18)^2+(0)^2+(36)^2}$ | | |
| | $\{=20.1246\ldots\}=9\sqrt{5}\ (\text{cm}^2)$ or awrt $20.1\ (\text{cm}^2)$ | A1 | 2.2a |
| | | (4) | |
| (a) Way 2 | Uses appropriate vectors to find an angle or perpendicular height in triangle $ABC$ and uses a correct method to make a complete attempt to find the area of triangle $ABC$. | M1 | 3.1b |
| | $\overrightarrow{AB}=\begin{pmatrix}4\\0\\-2\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}2\\9\\-1\end{pmatrix},\ \left\{\overrightarrow{BC}=\begin{pmatrix}-2\\9\\1\end{pmatrix}\right\}$ | M1 | 1.1b |
| | Uses a correct method to find an angle or perpendicular height in triangle $ABC$ | M1 | 1.1b |
| | **Note:** $B\hat{A}C=27.905\ldots,\ A\hat{B}C=76.047\ldots,\ B\hat{C}A=76.047\ldots$ or perpendicular height $=9$ | | |
| | Area $ABC=\frac{1}{2}\sqrt{86}\sqrt{20}\sin 76.047\ldots$ or $\frac{1}{2}\sqrt{86}\sqrt{86}\sin 27.905\ldots$ or $\frac{1}{2}\sqrt{20}(9)$ | | |
| | $\{=20.1246\ldots\}=9\sqrt{5}\ (\text{cm}^2)$ or awrt $20.1\ (\text{cm}^2)$ | A1 | 2.2a |
| | | (4) | |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (b) | Finds appropriate vectors to form the equation volume tetrahedron $ABCD=21$ to give a linear equation in $d$. **Note:** The volume must include $\frac{1}{6}$ | M1 | 3.1a |
| | e.g. $\left\lvert\begin{pmatrix}3\\7\\d-4\end{pmatrix}\bullet\begin{pmatrix}18\\0\\36\end{pmatrix}\right\rvert=\ldots$ or $\begin{vmatrix}4&0&-2\\2&9&-1\\3&7&d-4\end{vmatrix}=\ldots$ | M1 | 1.1b |
| | $=\lvert 54+36d-144\rvert$ or $\lvert 4(9d-36+7)-2(14-27)\rvert$ $\{=\lvert 36d-90\rvert\}$ | A1 | 1.1b |
| | $\left\{\frac{1}{6}\lvert 36d-90\rvert=21\Rightarrow\lvert 36d-90\rvert=126\Rightarrow\right\}\ d=6$ | A1 | 1.1b |
| | | (4) | |
| | | | (8 marks) |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $A(2,1,4)$, $B(6,1,2)$, $C(4,10,3)$, $D(5,8,d)$ | | |
| (a) Way 3 | Complete attempt to find the area of triangle $ABC$ by applying $\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB} + \overrightarrow{OB}\times\overrightarrow{OC} + \overrightarrow{OC}\times\overrightarrow{OA}\right\|$ or equivalent | M1 | 3.1b |
| | $\overrightarrow{OA}\times\overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 4 \\ 6 & 1 & 2 \end{vmatrix} = \dots$ and $\overrightarrow{OB}\times\overrightarrow{OC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 2 \\ 4 & 10 & 3 \end{vmatrix} = \dots$, and $\overrightarrow{OC}\times\overrightarrow{OA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 10 & 3 \\ 2 & 1 & 4 \end{vmatrix} = \dots$ | M1 | 1.1b |
| | $\{\overrightarrow{OA}\times\overrightarrow{OB} + \overrightarrow{OB}\times\overrightarrow{OC} + \overrightarrow{OC}\times\overrightarrow{OA}\} = \begin{pmatrix} -2 \\ 20 \\ -4 \end{pmatrix} + \begin{pmatrix} -17 \\ -10 \\ 56 \end{pmatrix} + \begin{pmatrix} 37 \\ -10 \\ -16 \end{pmatrix}$ | M1 | 1.1b |
| | Area $ABC = \frac{1}{2}\sqrt{(18)^2 + (0)^2 + (36)^2}$ | | |
| | $\{= 20.1246\dots\} = 9\sqrt{5}$ (cm$^2$) or awrt 20.1 (cm$^2$) | A1 | 2.2a |
| | | (4) | |

| Notes for Question | |
|---|---|
| (a) | Way 1 |
| M1: | Complete correct process of taking the vector product between 2 edges of triangle $ABC$, applying Pythagoras and multiplying the result by 0.5 |
| M1: | Uses a correct method to find any 2 edges of triangle $ABC$ |
| M1: | Attempts to take the vector cross product between 2 edges of triangle $ABC$ |
| A1: | Deduces the correct area of either $9\sqrt{5}$ (cm$^2$) or awrt 20.1 (cm$^2$) |
| (a) | Way 2 |
| M1: | See scheme |
| M1: | Uses a correct method to find any 2 edges of triangle $ABC$ |
| M1: | Either<br>• finds an angle in $ABC$ by using a correct scalar product method<br>• finds an angle in $ABC$ by using the cosine rule in the correct direction<br>• realises triangle $ABC$ is isosceles and applies Pythagoras in the correct direction to find the perpendicular height |
| A1: | Deduces the correct area as either $9\sqrt{5}$ (cm$^2$) or awrt 20.1 (cm$^2$) |
| Note: | For Way 1 and Way 2, using any of $\overrightarrow{OA}$, $\overrightarrow{OB}$ or $\overrightarrow{OC}$ in their vector product is M0 M0 A0 A0 |
| (a) | Way 3 |
| M1: | See scheme |
| M1: | Attempts to apply $\overrightarrow{OA}\times\overrightarrow{OB}$, $\overrightarrow{OB}\times\overrightarrow{OC}$ and $\overrightarrow{OC}\times\overrightarrow{OA}$ |
| A1: | Attempts to add (as vectors) the results of applying $\overrightarrow{OA}\times\overrightarrow{OB}$, $\overrightarrow{OB}\times\overrightarrow{OC}$ and $\overrightarrow{OC}\times\overrightarrow{OA}$ |
| A1: | Deduces the correct area as either $9\sqrt{5}$ (cm$^2$) or awrt 20.1 (cm$^2$) |

| | |
|---|---|
| **(b)** | |
| **M1:** | See scheme |
| **M1:** | Uses appropriate vectors in an attempt at the scalar triple product |
| **A1:** | Correct applied expression for the scalar triple product (allow $\pm$ and ignore modulus sign) |
| **A1:** | Correct solution leading to $d=6$ |
| **Note:** | Using any of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ or $\overrightarrow{OD}$ in their scalar triple product is M0 M0 A0 A0 |
| **Note:** | Some vector product calculations for reference: |
| | $\left\|\overrightarrow{AD}.\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right\|=\begin{vmatrix}3 & 7 & d-4\\ 4 & 0 & -2\\ 2 & 9 & -1\end{vmatrix}=\left\|\begin{pmatrix}3\\7\\d-4\end{pmatrix}\bullet\begin{pmatrix}18\\0\\36\end{pmatrix}\right\|=\|54+36d-144\|=\|36d-90\|$ |
| | $\left\|\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right\|=\begin{vmatrix}4 & 0 & -2\\ 2 & 9 & -1\\ 3 & 7 & d-4\end{vmatrix}=\left\|\begin{pmatrix}4\\0\\-2\end{pmatrix}\bullet\begin{pmatrix}9d-29\\5-2d\\-13\end{pmatrix}\right\|=\|36d-116+26\|=\|36d-90\|$ |
| | $\left\|\overrightarrow{AC}.\left(\overrightarrow{AB}\times\overrightarrow{AD}\right)\right\|=\begin{vmatrix}2 & 9 & -1\\ 4 & 0 & -2\\ 3 & 7 & d-4\end{vmatrix}=\left\|\begin{pmatrix}2\\9\\-1\end{pmatrix}\bullet\begin{pmatrix}14\\10-4d\\28\end{pmatrix}\right\|=\|28+90-36d-28\|=\|90-36d\|$ |

**Q4.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $\pm\overrightarrow{DE}=\pm\begin{pmatrix}-4\\3\\1\end{pmatrix},\ \pm\overrightarrow{DF}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix},\ \pm\overrightarrow{EF}=\pm\begin{pmatrix}3\\2\\1\end{pmatrix}$ | M1 | 1.1b |
| | $\text{Area}=\frac{1}{2}\left\|\overrightarrow{DE}\times\overrightarrow{DF}\right\|=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & 3 & 1\\ -1 & 5 & 2\end{vmatrix}=\frac{1}{2}\begin{vmatrix}1\\7\\-17\end{vmatrix}=\frac{1}{2}\sqrt{1^2+7^2+17^2}$ | M1 | 1.1b |
| | $=\frac{1}{2}\sqrt{339}(\text{cm}^2)$ * | A1* | 2.2a |
| | | (3) | |
| | **Alternative for (a) using trigonometry:** | | |
| | $\pm\overrightarrow{DE}=\pm\begin{pmatrix}-4\\3\\1\end{pmatrix},\ \pm\overrightarrow{DF}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix},\ \pm\overrightarrow{EF}=\pm\begin{pmatrix}3\\2\\1\end{pmatrix}$ | M1 | 1.1b |
| | $DE=\sqrt{26},\ DF=\sqrt{30},\ EF=\sqrt{14}$<br>$\cos DEF=\frac{26+14-30}{2\sqrt{26}\sqrt{14}}\Rightarrow DEF=\cos^{-1}\frac{5}{\sqrt{26}\sqrt{14}}$<br>$\Rightarrow \text{Area } DEF=\frac{1}{2}\times\sqrt{26}\sqrt{14}\sqrt{1-\frac{25}{364}}=\ldots$ | M1 | 1.1b |
| | $=\frac{1}{2}\times\sqrt{26}\sqrt{14}\frac{\sqrt{339}}{\sqrt{26}\sqrt{14}}=\frac{1}{2}\sqrt{339}$ * | A1 | 2.2a |

| | | | |
|---|---|---|---|
| **(b)** | Attempt to find "$T$", the 4th vertex of the tetrahedron e.g. $\begin{pmatrix}2\\-1\\8\end{pmatrix}+\lambda\overrightarrow{AD}=\begin{pmatrix}1\\4\\10\end{pmatrix}+\mu\overrightarrow{CF}\Rightarrow\lambda=\ldots$ or $\mu=\ldots$ <br> or e.g. <br> $DF=\sqrt{30},\ AC=\sqrt{120}\Rightarrow$ Linear SF $=2$ <br> $AT=2AD\Rightarrow T$ is ... | M1 | 3.1b |
| | $T(1, 1, 15)$ | A1 | 1.1b |
| | e.g $\overrightarrow{AT}=\begin{pmatrix}-2\\4\\14\end{pmatrix},\ \overrightarrow{BT}=\begin{pmatrix}6\\-2\\12\end{pmatrix},\ \overrightarrow{CT}=\begin{pmatrix}0\\-6\\10\end{pmatrix}$ <br> $\overrightarrow{AT}\times\overrightarrow{BT}\bullet\overrightarrow{CT}=\begin{vmatrix}-2&4&14\\6&-2&12\\0&-6&10\end{vmatrix}=\ldots$ | M1 | 1.1b |
| | e.g $V=\frac{1}{6}\left\|-2(-20+72)-4(60)+14(-36)\right\|\left(=\frac{424}{3}\right)$ | A1 | 1.1b |
| | $\frac{1}{6}\overrightarrow{DT}\times\overrightarrow{ET}\bullet\overrightarrow{FT}=\begin{vmatrix}-1&2&7\\3&-1&6\\0&-3&5\end{vmatrix}=\frac{1}{6}\left\|-1(13)-2(15)+7(-9)\right\|\left(=\frac{53}{3}\right)$ | M1 | 3.1b |
| | or <br> length scale factor $=2\Rightarrow$ volume scale factor $=8$ | | |
| | e.g Volume $=\frac{424}{3}-\frac{53}{3}=\ldots$ or Volume $=\frac{7}{8}\times\frac{424}{3}=\ldots$ <br> or Volume $=7\times\frac{53}{3}=\ldots$ | dM1 | 3.1a |
| | $=\frac{371}{3}\text{ cm}^3$ | A1 | 1.1b |
| | | (7) | |
| | See below for alternative for part (b) | | |
| | | | **(10 marks)** |

| Notes |
|---|
| (a)<br>M1: Attempts to find 2 edges of triangle *DEF*. Must be subtracting components.<br>M1: Uses the correct process of the vector product to attempt the area including use of Pythagoras<br>A1*: Deduces the correct area with no errors<br>Alternative:<br>M1: Attempts to find 3 edges of triangle *DEF*. Must be subtracting components.<br>M1: A complete method for the area. Allow work in decimals for this mark but must work in exact terms to obtain the A mark<br>A1*: Deduces the correct area with no errors and no decimal work<br>(b)<br>M1: Adopts a correct strategy to find the fourth vertex of the tetrahedron e.g. finding where two edges intersect or uses the linear scale factor<br>A1: Correct coordinates for the other vertex<br>M1: Uses the information from the design to attempt a scalar triple product between appropriate vectors to find the volume of the smaller or larger tetrahedron.<br>A1: Correct volume for either tetrahedron<br>M1: Makes further progress with the solution by finding the volume of the other tetrahedron or calculates the volume scale factor using an appropriate method. E.g. using ratios or by finding the area of triangle *DEF* and comparing with triangle *ABC*<br>dM1: Completes the problem by finding the required volume of the frustum. Depends on all previous method marks<br>A1: Correct answer |

| **Alternative for part (b) – splits into 4 tetrahedra:**<br>**This example takes *M* as the midpoint of *AC* and finds the volume of *ABMD, MBFC, DMFB, EDFB*** | | | |
|---|---|---|---|
| **(b)** | $Vol_{ABMD} = \frac{1}{6}\overrightarrow{AB}\times\overrightarrow{AD}\bullet\overrightarrow{AM} = \ldots$ | M1 | 3.1b |
| | $= \frac{106}{3}$ | A1 | 1.1b |
| | $Vol_{MBFC} = \frac{1}{6}\overrightarrow{BF}\times\overrightarrow{BC}\bullet\overrightarrow{BM} = \ldots\left(\frac{106}{3}\right)$<br>$Vol_{DMFB} = \frac{1}{6}\overrightarrow{DF}\times\overrightarrow{DM}\bullet\overrightarrow{DB} = \ldots\left(\frac{106}{3}\right)$ | M1<br>A1 | 1.1b<br>1.1b |
| | $Vol_{EDFB} = \frac{1}{6}\overrightarrow{ED}\times\overrightarrow{EF}\bullet\overrightarrow{EB} = \ldots\left(\frac{53}{3}\right)$ | M1 | 3.1b |
| | *ABMD* + *MBFC* + *DMFB* + *EDFB* = $3\times\frac{106}{3}+\frac{53}{3}$ | dM1 | 3.1a |
| | $= \frac{371}{3}\text{ cm}^3$ | A1 | 1.1b |

| Notes: |
|---|
| M1: Adopts a correct strategy to find the volume of one tetrahedron<br>A1: Any correct volume<br>M1: Adopts a correct strategy to find the volumes of at least 2 other tetrahedra<br>A1: Correct volumes<br>M1: Makes further progress with the solution by finding the volume of all relevant tetrahedra<br>dM1: Completes the problem by finding the required volume of the frustum. Depends on all previous method marks<br>A1: Correct answer |

**Q5.**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| | $\Pi_1: x-2y-3z=5,\quad \Pi_2: 6x+y-4z=7$ | | |
| **(a) Way 1** | $\begin{pmatrix}1\\-2\\-3\end{pmatrix}\cdot\begin{pmatrix}6\\1\\-4\end{pmatrix}=6-2+12$ | Attempts scalar product of normal vectors allowing one slip. May be implied by a value of 16. | M1 |
| | $16=\sqrt{1^2+2^2+3^2}\sqrt{6^2+1^2+4^2}\cos\theta$ $\Rightarrow \cos\theta=...$ | Complete attempt to find $\cos\theta$ | M1 |
| | $\cos\theta=\dfrac{16}{\sqrt{14}\sqrt{53}}\Rightarrow\theta=54°$ | Cao and do **not** isw. E.g. if they subsequently find 90 – 54 or 180 – 54, score A0. **Do not allow 54.0.** | A1 |
| **(a) Way 2** | $\begin{pmatrix}1\\-2\\-3\end{pmatrix}\times\begin{pmatrix}6\\1\\-4\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-2&-3\\6&1&-4\end{vmatrix}=\begin{pmatrix}11\\-14\\13\end{pmatrix}$ | Attempts cross product of normal vectors. 2 components should be correct if there is no working. | M1 |
| | $\sqrt{11^2+14^2+13^2}=\sqrt{1^2+2^2+3^2}\sqrt{6^2+1^2+4^2}\sin\theta$ $\Rightarrow\sin\theta=...$ | Complete attempt to find $\sin\theta$ | M1 |
| | $\sin\theta=\dfrac{9\sqrt{6}}{\sqrt{14}\sqrt{53}}\Rightarrow\theta=54°$ | Cao and do **not** isw. E.g. if they subsequently find 90 – 54 or 180 – 54, score A0. **Do not allow 54.0.** | A1 |
| | | | **(3)** |
| **(b)** | $\mathbf{PQ}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix}$ or $\begin{pmatrix}2+\lambda\\3-2\lambda\\-1-3\lambda\end{pmatrix}$ | Attempt parametric form of **PQ** by using the point $P$ and the normal to $\Pi_1$ | M1 |
| | $6(2+\lambda)+(3-2\lambda)-4(-1-3\lambda)=7$ $\Rightarrow\lambda=...$ | Substitutes parametric form of **PQ** into the equation of $\Pi_2$ and solves for $\lambda$ | M1 |
| | $\lambda=-\dfrac{3}{4}\Rightarrow Q\text{ is }\left(\dfrac{5}{4},\dfrac{9}{2},\dfrac{5}{4}\right)$ | M1: Uses their value of $\lambda$ in their **PQ** equation<br>A1: Correct coordinates or vector. | M1A1 |
| | | | **(4)** |

| | | | |
|---|---|---|---|
| **(c)** | $\begin{pmatrix}1\\-2\\-3\end{pmatrix}\times\begin{pmatrix}6\\1\\-4\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-2&-3\\6&1&-4\end{vmatrix}=\begin{pmatrix}11\\-14\\13\end{pmatrix}$ | M1: Attempt cross product between normals<br>A1: Correct normal vector (any multiple) | M1A1 |
| | Alternative:<br>$x-2y-3z=0,\ \ 6x+y-4z=0: x=1\Rightarrow y=-\dfrac{14}{11}, z=\dfrac{13}{11}$<br>$\Rightarrow\mathbf{n}=\begin{pmatrix}11\\-14\\13\end{pmatrix}$<br>M1: Solves $x-2y-3z=0,\ \ 6x+y-4z=0$ to obtain values for $x$, $y$ and $z$<br>A1: Correct vector (or values) | | |
| | $\begin{pmatrix}11\\-14\\13\end{pmatrix}\cdot\begin{pmatrix}\frac{5}{4}\\\frac{9}{2}\\\frac{5}{4}\end{pmatrix}=...$ or $\begin{pmatrix}11\\-14\\13\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix}=...$ | Attempt scalar product between their normal and their **OQ** or **OP**. Must obtain a value. | M1 |
| | $\mathbf{r}.\begin{pmatrix}11\\-14\\13\end{pmatrix}=-33$ | Any multiple e.g. $\mathbf{r}.\begin{pmatrix}11k\\-14k\\13k\end{pmatrix}=-33k\quad(k\neq0)$ | A1 |
| | Note that if they use the intersection with $\Pi_1$ $\left(\dfrac{17}{7},\dfrac{15}{7},\dfrac{-16}{7}\right)$ for $Q$ allow all the marks to score in (c). | | |
| | | | **(4)** |
| | | | **Total 11** |

**Q6.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Need **k** component to be zero at ground, so $0.84+0.8\lambda-\lambda^2=0 \Rightarrow \lambda=\ldots$ | M1 | 1.1b |
| | $\lambda=-\frac{3}{5},\frac{7}{5}$, but $\lambda \geqslant 0$ so $\lambda=\frac{7}{5}$ | A1 | 1.1b |
| | | (2) | |
| (b) | Direction is $(9-4.6\times1.4)\mathbf{i}+15\mathbf{j}+(0.8-2\times1.4)$ $=2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}$ or $\frac{64}{25}\mathbf{i}+15\mathbf{j}-2\mathbf{k}$ | B1ft | 2.2a |
| | | (1) | |

| | | | |
|---|---|---|---|
| (c) | Direction perpendicular to ground is $a\mathbf{k}$, so angle to perpendicular is given by $(\cos\theta)=\frac{a\mathbf{k}.(2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k})}{a\times\lvert 2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}\rvert}$ or $\frac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\0\\a\end{pmatrix}}{\begin{vmatrix}2.56\\15\\-2\end{vmatrix}\begin{vmatrix}0\\0\\a\end{vmatrix}}$ or angle between $\begin{pmatrix}2.56\\15\\-2\end{pmatrix}$ and $\begin{pmatrix}2.56\\15\\0\end{pmatrix}$ is given by $(\cos\theta)=\frac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}2.56\\15\\0\end{pmatrix}}{\begin{vmatrix}2.56\\15\\-2\end{vmatrix}\begin{vmatrix}2.56\\15\\0\end{vmatrix}}$ | M1 | 1.1b |
| | $=\frac{-2}{\sqrt{2.56^2+15^2+(-2)^2}}(=-0.130\ldots)$ Or $=\frac{231.5536}{\sqrt{2.56^2+15^2+(-2)^2}\sqrt{2.56^2+15^2+(0)^2}}=0.991\ldots$ | M1 | 1.1b |
| | $90°-\arccos('-0.130\ldots')=-7.48\ldots$ or $\arccos(0.991\ldots)$ | ddM1 | 3.1b |
| | So the tennis ball hits ground at angle of $7.5°$ (1d.p.) cao | A1 | 3.2a |
| | **Alternative** Finds the length of the vector in the **ij** plane $=\sqrt{2.56^2+15^2}$ | M1 | 1.1b |
| | $\tan\theta=\frac{2}{\sqrt{2.56^2+15^2}}$ | M1 | 1.1b |
| | $\theta=\arctan\left(\frac{2}{\sqrt{2.56^2+15^2}}\right)$ or $\theta=90-\arctan\left(\frac{\sqrt{2.56^2+15^2}}{2}\right)$ | ddM1 | 3.1b |

| | | | |
|---|---|---|---|
| | So the tennis ball hits ground at angle of $7.5°$ (1d.p.) | A1 | 3.2a |
| | | **(4)** | |
| **(d)** | In same plane as net when $\mathbf{r}.\mathbf{j}=0$, $\begin{pmatrix}-4.1+9\lambda-2.3\lambda^2\\-10.25+15\lambda\\0.84+0.8\lambda-\lambda^2\end{pmatrix}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}$ leading to $-10.25+15\lambda=0\Rightarrow\lambda=...$ $\left(=\frac{41}{60}=0.683333...\right)$ | M1 | 3.1b |
| | So is at position $\left(-4.1+9\times\frac{41}{60}-2.3\left(\frac{41}{60}\right)^2\right)\mathbf{i}+0\mathbf{j}+\left(0.84+0.8\times\frac{41}{60}-\left(\frac{41}{60}\right)^2\right)\mathbf{k}$ | M1 | 1.1b |
| | $=$ awrt $0.976\mathbf{i}+$ awrt $0.920\mathbf{k}$ or $=$ awrt $0.976\mathbf{i}+0.92\mathbf{k}$ (to 3 s.f.) or $=$ awrt $0.976\mathbf{i}+\frac{3311}{3600}\mathbf{k}$ | A1 | 1.1b |
| | | **(3)** | |
| **(e)** | Modelling as a line, height of net is 0.9m along its length so as **0.92 > 0.9** the ball **will** pass over the net according to the model. | B1ft | 3.2a |
| | | **(1)** | |
| **(f)** | Identifies a suitable feature of the model that affects the outcome<br>And uses it to draw a compatible conclusion.<br>For example<br>• The ball is **not a particle** and will have **diameter/radius**, therefore it will hit the net and not pass over.<br>• As above, but so the **ball will clip the net** but it's momentum will take it over as it is mostly above the net.<br>• The **model** says that the ball will clear the net by 2cm which may be smaller than the balls **diameter**<br>• The **net will not be a straight line/taut** so will not be 0.9m high, so the ball **will** have enough clearance to pass over the net. | M1<br>A1 | 3.2b<br>2.2b |
| | | **(2)** | |
| | | | **(13 marks)** |

**Notes:**

**Accept any alternative vector notations throughout.**

**(a)**

**M1:** Attempts to solve the quadratic from equating the **k** component to zero.

**A1:** Correct value, must select positive root, so accept 1.4 oe.

Correct answer only M1 A1

**(b)**

**B1ft:** For $(2.56,15,-2)$ o.e or follow through $(9-4.6\times'\lambda',15,0.8-2\times'\lambda')$ for their $\lambda$.

**(c)**

**M1:** Recognises the angle between the perpendicular and direction vector is needed, and identifies the perpendicular as $a\mathbf{k}$ for any non-zero $a$ (including 1), and attempts dot product

Alternatively recognises the dot product of $(2.56, 15, -2)$ and $(2.56, 15, 0)$

**M1:** Applies the dot product formula $\frac{a \bullet b}{|a||b|}$ correctly between *any* two vectors, but must have dot product and modulus evaluated.

**ddM1:** Dependent on both previous marks. A correct method to proceed to the required angle, usually $90° - \arccos('-0.130...')$ as shown in scheme but may e.g. use $\sin\theta$ instead of $\cos\theta$ in formula.

Alternatively is using dot product of $(2.56, 15, -2)$ and $(2.56, 15, 0)$ finds arccos(0.991…)

**A1:** For 7.5° cao

**Alternative**

**M1:** Finds the length of the vector in the **ij** plane.

**M1:** Finds the tan of any angle the

**ddM1:** Dependent on both previous marks. Finds the required angle

**A1:** For 7.5° cao

**(d)**

**M1:** Attempts to find value of $\lambda$ that gives zero **j** component.

**M1:** Uses their value of $\lambda$ in the equation of the path to find position.

**A1:** Correct position.

**(e)**

**B1ft:** States that 0.920 > 0.9 so according to the model the ball will pass over the net. Follow through on their **k** component and draws an appropriate conclusion. May stay the value of k > 0.92

**(f)**

**M1:** There must be some reference to the model to score this mark. See scheme for examples. It is likely to be either the ball is not a particle, or the top of the net is not a straight line. Accept references to the ball crossing a long way from the middle.

Do not accept reasons such as "there may be wind/air resistance" as these are not referencing the given model.

**A1:** For a reasonable conclusion based on their reference to the model.

**For example**

The ball is not a particle; therefore, it will not go over the net is M1A0 as not explained why – needs reference to radius/diameter

**Q7.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **(a)** | $\mathbf{n} = \begin{pmatrix}1\\-2\\25\end{pmatrix}$ or any non-zero scalar multiple thereof | **B1** | 1.2 |
| | | **(1)** | |
| **(b)** | $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 25\\ 3 & -2 & -1\end{vmatrix} = \begin{pmatrix}(-2)(-1)-(-2)(25)\\ -((1)(-1)-(3)(25))\\ (1)(-2)-(3)(-2)\end{pmatrix} = \ldots$ | **M1** | 1.1b |
| | $= \begin{pmatrix}52\\76\\4\end{pmatrix} = 4\begin{pmatrix}13\\19\\1\end{pmatrix}$ (or correct multiple for their normal vector used) | **A1** | 2.1 |
| | | **(2)** | |
| **(c)** | Landing direction is perpendicular to $\mathbf{n}\times\mathbf{v}_\mathbf{A}$ and $\mathbf{n}$ so required direction is given by $\begin{pmatrix}13\\19\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\25\end{pmatrix} = \ldots$<br>**Alternatively** recognises the recognises the landing direction is the line of intersection of the plane containing $\mathbf{n}$ and $\mathbf{v}_\mathbf{A}$ and the plane representing the field.<br>Finds the equation of the plane containing $\mathbf{n}$ and $\mathbf{v}_\mathbf{A}$<br>$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}13\\19\\1\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix}\begin{pmatrix}13\\19\\1\end{pmatrix}$<br>$x - 2y + 25z = 0$ and $13x + 19y + z = 0$ | **M1** | 3.1b |
| | $= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 13 & 19 & 1\\ 1 & -2 & 25\end{vmatrix} = \begin{pmatrix}(19)(25)-(-2)(1)\\ -((13)(25)-(1)(1))\\ (13)(-2)-(1)(19)\end{pmatrix} = \ldots$<br>**Alternative**<br>Selects a value for either $x$, $y$ or $z$ and solves simultaneously<br>e.g $z = -5$ leading to<br>$x - 2y = 125$ and $13x + 19y = 5 \Rightarrow x = \ldots,\ y = \ldots$ | **M1** | 3.4 |
| | $= \begin{pmatrix}477\\-324\\-45\end{pmatrix}$ or any positive multiple thereof, e.g $\begin{pmatrix}53\\-36\\-5\end{pmatrix}$ or $\begin{pmatrix}1908\\-1296\\-180\end{pmatrix}$ | **A1** | 1.1b |
| | | **(3)** | |
| **(d)** | Any acceptable reason e.g<br>- Paths would not be linear<br>- May have some lateral movement | **B1** | 3.5b |

| | | | |
|---|---|---|---|
| | - Could be affected by cross winds<br>- Field might not be flat | | |
| | | **(1)** | |
| | | | **(7 marks)** |

**Notes:**

**(a)**

**B1:** Correct normal vector (any non-zero scalar multiple thereof is fine)

**(b)**

**M1:** Uses their $\mathbf{n}$ and $\mathbf{v}_A$ in cross product formula. Allow slips in coordinates as long as the intent is clear.

**A1:** Correct work leading to a multiple of the required vector – may be a different multiple to the one shown if their $\mathbf{n}$ was different.

**(c)**

**M1:** Uses a correct strategy to find the direction, ie realises $\mathbf{v}_A$ must be perpendicular to both the vector from (b) and $\mathbf{n}$. Allow for vectors used either way round.

Other methods may be possible – e.g finds the plane containing $\mathbf{n}$ and $\mathbf{v}_A$ and solves with the plane representing the field.

**M1:** Uses their answer to (b) with the normal vector to find a vector in the direction required. Allow for vectors either way round.

Alternative find the line of intersection of the plane containing $\mathbf{n}$ and $\mathbf{v}_A$ and the field.

**A1:** A correct direction vector, as shown or any positive multiple. For this mark the direction should be correct – so order of vectors must have been correct, or adapted to correct direction if initially incorrect.

**(d)**

**B1:** Any correct limitation given. See scheme for examples.

**Q8.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | E.g. $\overrightarrow{AB}=\begin{pmatrix}-25\\9\\5\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}-20\\5\\-4\end{pmatrix}$ | M1 | 1.1b |
| | $\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-25&9&5\\-20&5&-4\end{vmatrix}=\ldots$ | M1 | 1.1b |
| | $\text{Area}=\frac{1}{2}\begin{vmatrix}-61\\-200\\55\end{vmatrix}=\frac{1}{2}\sqrt{61^2+200^2+55^2}=108*$ | A1* | 2.2a |
| | | (3) | |
| | **(a) Alternative:** | | |
| | $AB=\sqrt{25^2+9^2+5^2},\ AC=\sqrt{20^2+5^2+4^2},\ BC=\sqrt{5^2+4^2+9^2}$ | M1 | 1.1b |
| | $AC^2=AB^2+BC^2-2AB\times BC\cos ABC$<br>$\Rightarrow 441=731+122-2\times\sqrt{731}\times\sqrt{122}\cos ABC$<br>$\Rightarrow \cos ABC=\frac{731+122-441}{2\times\sqrt{731}\times\sqrt{122}}$ | M1 | 1.1b |
| | $\text{Area}=\frac{1}{2}\sqrt{731}\sqrt{122}\sin ABC=108*$ | A1 | 2.2a |

| | | | |
|---|---|---|---|
| (b) | A complete attempt to find the volume of the tetrahedron | M1 | 3.1a |
| | E.g. $\begin{vmatrix} 18 & -14 & -2 \\ -7 & -5 & 3 \\ -2 & -9 & -6 \end{vmatrix} = \ldots$ | M1 | 1.1b |
| | $= 1592$ | A1 | 1.1b |
| | $V = \dfrac{1592}{6}$ or e.g. $V = \dfrac{796}{3}$ | A1 | 1.1b |
| | | **(4)** | |
| (c) | $\text{Mass} = \dfrac{1592}{6} \times 0.85 \div 1000 \ \ (\text{kg})$ | M1 | 2.1 |
| | $\{= 0.2255333\ldots\} =$ awrt $0.226$ (kg) | A1 | 1.1b |
| | | **(2)** | |
| | | | **(9 marks)** |

**Notes**

(a)
M1: Attempts to find 2 edges of the required triangle
M1: Uses the correct process of the vector product for 2 appropriate vectors
A1*: Deduces the correct area with no errors but condone sign slips on the components provided the work is otherwise correct e.g. allow $\text{Area} = \dfrac{1}{2}\begin{vmatrix} -61 \\ -200 \\ -55 \end{vmatrix} = \dfrac{1}{2}\sqrt{61^2 + 200^2 + 55^2} = 108*$

**Alternative**
M1: Attempts lengths of all 3 sides
M1: Applies the cosine rule to find one of the angles of the triangle
A1*: Deduces the correct area with no errors
(b)
M1: See scheme
M1: Uses appropriate vectors in an attempt at the scalar triple product
A1: Correct numerical expression for the scalar triple product (allow $\pm$)
A1: Correct volume
(c)
M1: A correct method for changing their units for their volume and for finding the mass in kg
A1: Correct answer

**Q9.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | A correct method to find one coordinates of $M$, $N$ or $P$<br>For example<br>$\overrightarrow{AB}=\begin{pmatrix}6\\-6\\6\end{pmatrix}$ so $\overrightarrow{OM}=\begin{pmatrix}3\\2\\-4\end{pmatrix}+\frac{4}{6}\begin{pmatrix}6\\-6\\6\end{pmatrix}=\ldots$<br>$\overrightarrow{AC}=\begin{pmatrix}-9\\-12\\12\end{pmatrix}$ so $\overrightarrow{ON}=\begin{pmatrix}3\\2\\-4\end{pmatrix}+\frac{4}{12}\begin{pmatrix}-9\\-12\\12\end{pmatrix}=\ldots$<br>$\overrightarrow{AD}=\begin{pmatrix}-7\\-7\\14\end{pmatrix}$ so $\overrightarrow{OP}=\begin{pmatrix}3\\2\\-4\end{pmatrix}+\frac{4}{14}\begin{pmatrix}-7\\-7\\14\end{pmatrix}=\ldots$ | M1 | 3.1a |
| | One of $(M=)(7,-2,0)$, $(N=)(0,-2,0)$ or $(P=)(1,0,0)$ | A1 | 1.1b |
| | All of $(M=)(7,-2,0)$, $(N=)(0,-2,0)$ and $(P=)(1,0,0)$ | A1 | 1.1b |
| | | (3) | |
| (b) | Correct method, e.g. realises $MN$ is parallel to $x$ axis, so base is 7 and height 2, hence area of intersection is $\frac{1}{2}\times7\times2=\ldots$<br>**Alternatively using** $\frac{1}{2}\lvert a\times b\rvert$<br>$\overrightarrow{PM}=\pm\begin{pmatrix}6\\-2\\0\end{pmatrix}$ $\overrightarrow{PN}=\pm\begin{pmatrix}-1\\-2\\0\end{pmatrix}$ $\overrightarrow{NM}=\pm\begin{pmatrix}7\\0\\0\end{pmatrix}$<br>For example $\frac{1}{2}\left\lvert\overrightarrow{MP}\times\overrightarrow{PN}\right\rvert=\frac{1}{2}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\6&-2&0\\-1&-2&0\end{vmatrix}=\frac{1}{2}\left\lvert\begin{pmatrix}0\\0\\-14\end{pmatrix}\right\rvert=\ldots$ | M1 | 1.1b |
| | $=7$ cso | A1 | 1.1b |
| | | (2) | |

| | | | |
|---|---|---|---|
| (c) | Vol $NMPA=\frac{1}{3}A_bh=\frac{1}{3}\times7\times4=\frac{28}{3}$<br>Or using triple scalar product<br>$NMPA=\frac{1}{6}\left\lvert\overrightarrow{AM}\cdot\left(\overrightarrow{AN}\times\overrightarrow{AP}\right)\right\rvert=\frac{1}{6}\left\lvert\begin{pmatrix}4\\-4\\4\end{pmatrix}\cdot\left(\begin{pmatrix}-3\\-4\\4\end{pmatrix}\times\begin{pmatrix}-2\\-2\\4\end{pmatrix}\right)\right\rvert$<br>$=\frac{1}{6}\left\lvert\begin{pmatrix}4\\-4\\4\end{pmatrix}\cdot\begin{pmatrix}-8\\4\\-2\end{pmatrix}\right\rvert=\frac{28}{3}$ | M1<br>A1 | 3.1a<br>1.1b |

| | | | |
|---|---|---|---|
| | Vol $ABCD=\frac{1}{6}\left\|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right\|=...$ <br> $=\frac{1}{6}\left\|\begin{pmatrix}6\\-6\\6\end{pmatrix}\cdot\left(\begin{pmatrix}-9\\-12\\12\end{pmatrix}\times\begin{pmatrix}-7\\-7\\14\end{pmatrix}\right)\right\|=\frac{1}{6}\left\|\begin{pmatrix}6\\-6\\6\end{pmatrix}\cdot\begin{pmatrix}-84\\42\\-21\end{pmatrix}\right\|=...$ | M1 | 1.1b |
| | $=147$ | A1 | 1.1b |
| | So volume required is $'147'-\frac{'28'}{3}=...$ | M1 | 3.1a |
| | $=\frac{413}{3}$ | A1 | 1.1b |
| | | (6) | |
| | | | (11 marks) |

**Notes:**

(a)

**M1**: Correct method for finding at least one of the three points. Allow one slip in coordinates but should have correct fraction to make the value of $z$ to be 0.

**A1**: Any one of the three points correct, ignoring the labelling.

**A1**: All three points correct, ignoring the labelling

(b)

**M1**: Correct method for finding the area of the triangle, e.g realises that $MN$ is parallel to the $x$-axis so uses $\frac{1}{2}bh$ with $b=MN$ and $h$ is distance of $MN$ from axis.

Alternative using $\frac{1}{2}|a\times b|$ with vectors $\overrightarrow{PM}=\pm\begin{pmatrix}6\\-2\\0\end{pmatrix}$ $\overrightarrow{PN}=\pm\begin{pmatrix}-1\\-2\\0\end{pmatrix}$ $\overrightarrow{NM}=\pm\begin{pmatrix}7\\0\\0\end{pmatrix}$ follow through on their answers in part (a). Condone sign slips except they must be using $-\mathbf{j}$ in the cross product

For example $\frac{1}{2}\left|\overrightarrow{MP}\times\overrightarrow{PN}\right|=\frac{1}{2}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\6&-2&0\\-1&-2&0\end{vmatrix}=\frac{1}{2}\left|\begin{pmatrix}0\\0\\-14\end{pmatrix}\right|=...$

$$\frac{1}{2}\left|\overrightarrow{PN}\times\overrightarrow{NM}\right|=\frac{1}{2}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&-2&0\\7&0&0\end{vmatrix}=\frac{1}{2}\left|\begin{pmatrix}0\\0\\14\end{pmatrix}\right|=...$$

$$\frac{1}{2}\left|\overrightarrow{PM}\times\overrightarrow{NM}\right|=\frac{1}{2}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\6&-2&0\\7&0&0\end{vmatrix}=\frac{1}{2}\left|\begin{pmatrix}0\\0\\14\end{pmatrix}\right|=...$$

or attempting to find an angle using dot product or cosine rule followed by $\frac{1}{2}ab\sin C$.

**A1**: Correct area of 7 from correct vectors

**(c) On ePen this is M1 A1 M1 M1 M1 A1**

**M1:** Formulates a correct method to find the volume of *NMPA*. May use method shown, or e.g. $\frac{1}{6}\left|\overrightarrow{AM}\cdot\left(\overrightarrow{AN}\times\overrightarrow{AP}\right)\right|$ or equivalent method.

**A1:** For $\frac{28}{3}$.

Note there are many ways to find the required volume of *AMNP* applying the triple scalar product to a combination of the following vectors

$$\overrightarrow{AM}=\begin{pmatrix}4\\-4\\4\end{pmatrix}\ \overrightarrow{AN}=\begin{pmatrix}-3\\-4\\4\end{pmatrix}\ \overrightarrow{AP}=\begin{pmatrix}-2\\-2\\4\end{pmatrix}\ \overrightarrow{NA}=\begin{pmatrix}3\\4\\-4\end{pmatrix}\ \overrightarrow{NM}=\begin{pmatrix}7\\0\\0\end{pmatrix}\ \overrightarrow{NP}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$\overrightarrow{MA}=\begin{pmatrix}-4\\4\\-4\end{pmatrix}\ \overrightarrow{MN}=\begin{pmatrix}-7\\0\\0\end{pmatrix}\ \overrightarrow{MP}=\begin{pmatrix}-6\\2\\0\end{pmatrix}\ \overrightarrow{PA}=\begin{pmatrix}2\\2\\-4\end{pmatrix}\ \overrightarrow{PM}=\begin{pmatrix}6\\-2\\0\end{pmatrix}\ \overrightarrow{PN}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$$

For example

$$\frac{1}{6}\left|\overrightarrow{AM}\cdot\left(\overrightarrow{AN}\times\overrightarrow{AP}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}4\\-4\\4\end{pmatrix}\cdot\left(\begin{pmatrix}-3\\-4\\4\end{pmatrix}\times\begin{pmatrix}-2\\-2\\4\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}4\\-4\\4\end{pmatrix}\cdot\begin{pmatrix}-8\\4\\-2\end{pmatrix}\right|=\frac{1}{6}\times 56$$

$$\frac{1}{6}\left|\overrightarrow{NA}\cdot\left(\overrightarrow{NM}\times\overrightarrow{NP}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}3\\4\\-4\end{pmatrix}\cdot\left(\begin{pmatrix}7\\0\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}3\\4\\-4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\14\end{pmatrix}\right|=\frac{1}{6}\times 56$$

$$\frac{1}{6}\left|\overrightarrow{MA}\cdot\left(\overrightarrow{MN}\times\overrightarrow{MP}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}-4\\4\\-4\end{pmatrix}\cdot\left(\begin{pmatrix}-7\\0\\0\end{pmatrix}\times\begin{pmatrix}-6\\2\\0\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}-4\\4\\-4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\14\end{pmatrix}\right|=\frac{1}{6}\times 56$$

$$\frac{1}{6}\left|\overrightarrow{PA}\cdot\left(\overrightarrow{PM}\times\overrightarrow{PN}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}2\\2\\-4\end{pmatrix}\cdot\left(\begin{pmatrix}6\\2\\0\end{pmatrix}\times\begin{pmatrix}-1\\-2\\0\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}2\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-14\end{pmatrix}\right|=\frac{1}{6}\times 56$$

Note candidates may write as $\frac{1}{6}\begin{vmatrix}4&-4&4\\-3&-4&4\\-2&-2&4\end{vmatrix}=\frac{1}{6}\left|4(-16+8)+4(-12+8)+4(6-8)\right|=\frac{1}{6}\left|-56\right|=\frac{28}{3}$

**M1:** A complete attempt at the volume of *ABCD*, with correct method for cross product (oe in other methods). Condone sign slips except they must be using $-\mathbf{j}$ in the cross product

**A1 (M1 on ePen):** 147

**M1:** Finds difference of the two volumes must have used a correct method to find the volumes.

**A1:** $\frac{413}{3}$

Note there are many ways to find the required volume of *ABCD* applying the triple scalar product to a combination of the following vectors

$$\overrightarrow{AB}=\begin{pmatrix}6\\-6\\6\end{pmatrix}\ \overrightarrow{AC}=\begin{pmatrix}-9\\-12\\12\end{pmatrix}\ \overrightarrow{AD}=\begin{pmatrix}-7\\-7\\14\end{pmatrix}\ \overrightarrow{BA}=\begin{pmatrix}-6\\6\\-6\end{pmatrix}\ \overrightarrow{BC}=\begin{pmatrix}15\\-6\\6\end{pmatrix}\ \overrightarrow{BD}=\begin{pmatrix}-13\\-1\\8\end{pmatrix}$$

$$\overrightarrow{CA}=\begin{pmatrix}9\\12\\-12\end{pmatrix}\ \overrightarrow{CB}=\begin{pmatrix}-15\\6\\-6\end{pmatrix}\ \overrightarrow{CD}=\begin{pmatrix}2\\5\\2\end{pmatrix}\ \overrightarrow{DA}=\begin{pmatrix}7\\7\\-14\end{pmatrix}\ \overrightarrow{DB}=\begin{pmatrix}13\\1\\-8\end{pmatrix}\ \overrightarrow{DC}=\begin{pmatrix}-2\\-5\\2\end{pmatrix}$$

For example

$$\frac{1}{6}\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}6\\-6\\6\end{pmatrix}\cdot\left(\begin{pmatrix}-9\\-12\\12\end{pmatrix}\times\begin{pmatrix}-7\\-7\\14\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}6\\-6\\6\end{pmatrix}\cdot\begin{pmatrix}-84\\42\\-21\end{pmatrix}\right|=\frac{1}{6}\times 882=147$$

$$\frac{1}{6}\left|\overrightarrow{BA}\cdot\left(\overrightarrow{BC}\times\overrightarrow{BD}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}-6\\6\\-6\end{pmatrix}\cdot\left(\begin{pmatrix}15\\-6\\6\end{pmatrix}\times\begin{pmatrix}-13\\-1\\8\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}-6\\6\\-6\end{pmatrix}\cdot\begin{pmatrix}-42\\42\\-63\end{pmatrix}\right|=\frac{1}{6}\times 882=147$$

$$\frac{1}{6}\left|\overrightarrow{CA}\cdot\left(\overrightarrow{CD}\times\overrightarrow{CB}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}9\\12\\-12\end{pmatrix}\cdot\left(\begin{pmatrix}2\\5\\2\end{pmatrix}\times\begin{pmatrix}15\\6\\-6\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}9\\12\\-12\end{pmatrix}\cdot\begin{pmatrix}-42\\42\\-63\end{pmatrix}\right|=\frac{1}{6}\times 882=147$$

$$\frac{1}{6}\left|\overrightarrow{DA}\cdot\left(\overrightarrow{DB}\times\overrightarrow{DC}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}7\\7\\-14\end{pmatrix}\cdot\left(\begin{pmatrix}13\\1\\-8\end{pmatrix}\times\begin{pmatrix}-2\\-5\\2\end{pmatrix}\right)\right|=\frac{1}{6}\left|\begin{pmatrix}7\\7\\-14\end{pmatrix}\cdot\begin{pmatrix}42\\-42\\63\end{pmatrix}\right|=\frac{1}{6}\times 882=147$$

Note candidates may write as

$$\frac{1}{6}\begin{vmatrix}6&-6&6\\-9&-12&12\\-7&-7&14\end{vmatrix}=\frac{1}{6}\left|6(-168+84)+6(-126+84)+6(63-84)\right|=\frac{1}{6}\left|-882\right|=147$$

**Q10.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | A complete method to use the scalar product of the direction vectors and the angle 120° to form an equation in $a$ $$\frac{\begin{pmatrix}2\\a\\0\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}}{\sqrt{2^2+a^2}\sqrt{1^2+(-1)^2}}=\cos 1\,20$$ | M1 | 3.1b |
| | $$\frac{a}{\sqrt{4+a^2}\sqrt{2}}=-\frac{1}{2}$$ | A1 | 1.1b |
| | $2a=-\sqrt{4+a^2}\sqrt{2}\Rightarrow 4a^2=8+2a^2\Rightarrow a^2=4\Rightarrow a=\ldots$ | M1 | 1.1b |
| | $a=-2$ | A1 | 2.2a |
| | | **(4)** | |
| (b) | Any two of **i**: $-1+2\lambda=4$ (1)<br>**j**: $5+\text{'their}-2\text{'}\lambda=-1+\mu$ (2)<br>**k**: $2=3-\mu$ (3) | M1 | 3.4 |
| | Solves the equations to find a value of $\lambda\left\{=\frac{5}{2}\right\}$ and $\mu\{=1\}$ | M1 | 1.1b |
| | $r_1=\begin{pmatrix}-1\\5\\2\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\\text{'their}-2\text{'}\\0\end{pmatrix}$ or $r_2=\begin{pmatrix}4\\-1\\3\end{pmatrix}+1\begin{pmatrix}0\\1\\-1\end{pmatrix}$ | dM1 | 1.1b |
| | $(4,0,2)$ or $\begin{pmatrix}4\\0\\2\end{pmatrix}$ | A1 | 1.1b |
| | Checks the third equation e.g.<br>$\lambda=\frac{5}{2}$: L **HS** $=5-2\lambda=5-5=0$<br>$\mu=1$: R **HS** $=-1+\mu=-1+1=0$<br>therefore **common point/intersect/consistent/tick**<br>or substitutes the values of $\lambda$ and $\mu$ into the relevant lines and achieves the same coordinate | B1 | 2.1 |
| | | **(5)** | |

| | | | |
|---|---|---|---|
| (c) | Full attempt to find the minimum distance from the point of intersection (nest) to the plane (ground)<br>E.g. Minimum distance $=\frac{\lvert 2\times\text{'4'}+(-3)\times\text{'0'}+1\times\text{'2'}-2\rvert}{\sqrt{2^2+(-3)^2+1)^2}}=\ldots$<br>**Alternatively**<br>$\mathbf{r}=\begin{pmatrix}\text{'4'}\\\text{'0'}\\\text{'2'}\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}$ $2(\text{'4'}+2\lambda)-3(\text{'0'}-3\lambda)+(\text{'2'}+\lambda)=2\Rightarrow$<br>$\lambda=\ldots\left\{-\frac{4}{7}\right\}$ | M1<br><br><br><br><br>A1ft | 3.1b<br><br><br><br><br>3.4 |

| | | | |
|---|---|---|---|
| | $\mathbf{r} = \begin{pmatrix} '4' \\ '0' \\ '2' \end{pmatrix} + ' - \frac{4}{7}' \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{7} \\ \frac{12}{7} \\ \frac{10}{7} \end{pmatrix}$ <br> Minimum distance $= \sqrt{\left(2 \times -\frac{4}{7}'\right)^2 + \left(-3 \times -\frac{4}{7}'\right)^2 + \left(1 \times -\frac{4}{7}\right)^2} =$ … <br> $= \sqrt{\left('4' - '\frac{20}{7}'\right)^2 + \left('0' - '\frac{12}{7}'\right)^2 + \left('2' - '\frac{10}{7}'\right)^2} = \ldots$ | | |
| | $\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$ or awrt 2.1 | A1 | 2.2b |
| | | **(3)** | |
| | **Alternative** <br> Find perpendicular distance from plane to the origin $2x - 3y + z = 2$ $\lvert n \rvert = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$ shortest distance $= \frac{2}{\sqrt{14}}$ <br> Find perpendicular distance from the plane containing the point of intersection to the origin $2x - 3y + z = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 10$ shortest distance $= \frac{10}{\sqrt{14}}$ <br> Minimum distance $= \frac{10}{\sqrt{14}} - \frac{2}{\sqrt{14}}$ | M1 <br> A1ft | 3.1b <br> 3.4 |
| | $\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$ or awrt 2.1 | A1 | 2.2b |
| | | **(3)** | |
| **(d)** | For example <br> Not reliable as the birds will not fly in a straight line <br> Not reliable as angle between flights paths will not always be 120° <br> Not reliable/reliable as the ground will not be flat/smooth <br> Not reliable as bird's nest is not a point | B1 | 3.2b |
| | | **(1)** | |
| | | | **(13 marks)** |

| |
|---|
| **Notes:** |
| **(a)** <br> **M1:** See scheme, allow a sign slip and cos 60 <br> **A1:** Correct simplified equation in $a$, cos 120 must be evaluated to – ½ and dot product calculated <br> **Note:** If the candidate states either $\left\lvert \frac{a \cdot b}{\lvert a \rvert \lvert b \rvert} \right\rvert = \cos\theta$ or $\left\lvert \frac{a}{\sqrt{4+a^2}\sqrt{2}} \right\rvert = \cos 60$ then has the equation $\frac{a}{\sqrt{4+a^2}\sqrt{2}} = \frac{1}{2}$ award this mark. If the module of the dot product is not seen then award A0 for this equation. |

| |
|---|
| **dM1:** Solve a quadratic equation for $a$, by squaring and solving an equation of the form $a^2 = K$ where $K > 0$<br>**A1:** Deduces the correct value of $a$ from a correct equation. Must be seen in part (a) using the angle between the lines.<br>**Alternative cross product method**<br>**M1:** $\begin{vmatrix} 2 & a & 0 \\ 0 & 1 & -1 \end{vmatrix} = \sqrt{2^2 + a^2}\sqrt{1^2 + (-1)^2}\, sin\, 1\,20$<br>**A1:** $\sqrt{a^2 + 8} = \sqrt{4 + a^2}\sqrt{2}\frac{\sqrt{3}}{2}$<br>Then as above<br>**Note If they use the point of intersection to find a value for $a$ this scores no marks** |
| **(b)**<br>**M1:** Uses the model to write down any two correct equations<br>**M1:** Solve two equations simultaneously to find a value for $\mu$ and $\lambda$<br>**dM1**: Dependent on previous method mark. Substitutes $\mu$ and $\lambda$ into a relevant equation. If no method shown two correct ordinates implies this mark.<br>**A1:** Correct coordinates. May be seen in part (c)<br>**B1**: Shows that the values of $\mu$ and $\lambda$ give the same third coordinate or point of intersection and draws the conclusion that the **lines intersect/common point/consistent** or tick.<br>**Note:** If an incorrect value for $a$ is found in part (a) but in part (b) they find that $a = -2$ this scores **B0** but all other marks are available |

| |
|---|
| **(c) This is M1M1A1 on ePen marking as M1 A1ft A1**<br>**M1:** Full attempt to find the minimum distance from a point to a plane. Condone a sign slip with the value of $d$.<br>**A1ft:** Following through on their point of intersection. Uses the model to find a correct expression for minimum distance from the nest to the ground<br>**A1:** Correct distance<br><br>**Alternative**<br>**M1:** Find the shortest distance from a point to plane by finding the perpendicular distance from the given plane to the origin and the perpendicular distance from the plane contacting their point of intersection to the origin and subtracts<br>**A1ft:** Following through on their point of intersection. Uses the model to find a correct expression for minimum distance from the nest to the ground<br>**A1:** Correct distance |
| **(d)**<br>**B1:** Comments on one of the models<br>• Flight path of the birds modelled as a straight line<br>• Angle between flight paths modelled as 120°<br>• The bird's nest is modelled as a point<br>• Ground modelled as a plane<br>Then states unreliabl<br>Any correct answer seen, ignore any other incorrect answers |

**Q11.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **Way 1** | Position of $A$ is given by $\overrightarrow{OA}=\begin{pmatrix}12+9\lambda\\16+6\lambda\\-8+2\lambda\end{pmatrix}$ | **B1** | 3.1a |
| | So have $\dfrac{12+9\lambda}{\sqrt{(12+9\lambda)^2+(16+6\lambda)^2+(-8+2\lambda)^2}}=\dfrac{3}{7}$ | **M1** | 1.1b |
| | $\Rightarrow 49\left(3\left(4+3\lambda\right)\right)^2=9\left((12+9\lambda)^2+(16+6\lambda)^2+(-8+2\lambda)^2\right)$<br>$\Rightarrow 2880\lambda^2+7200\lambda+2880=0$ or $2\lambda^2+5\lambda+2=0$ | **M1**<br>**A1** | 3.1a<br>2.1 |
| | $\Rightarrow(2\lambda+1)(\lambda+2)=0\Rightarrow\lambda=\ldots$ | **M1** | 1.1b |
| | Substitutes a value of $\lambda$ to find a position for A<br>e.g. $\overrightarrow{OA}=\begin{pmatrix}12+9\left(-\frac{1}{2}\right)\\16+6\left(-\frac{1}{2}\right)\\-8+2\left(-\frac{1}{2}\right)\end{pmatrix}=\ldots$ | **M1** | 1.1b |
| | Coordinates of $A$ are $\left(\dfrac{15}{2},13,-9\right)$ only | **A1** | 2.3 |
| | | (7) | |

| | | | |
|---|---|---|---|
| **Way 2** | Direction of $\overrightarrow{OA}$ is given by $\mathbf{d}=\begin{pmatrix}\frac{3}{7}k\\\beta k\\\gamma k\end{pmatrix}$ or use of $\left(\dfrac{3}{7}\right)^2+\beta^2+\gamma^2=1$ | **B1** | 3.1a |
| | $\begin{pmatrix}\frac{3}{7}k-12\\\beta k-16\\\gamma k+8\end{pmatrix}\times\begin{pmatrix}9\\6\\2\end{pmatrix}=0\Rightarrow\begin{cases}6\left(\frac{3}{7}k-12\right)-9(\beta k-16)=0\\2\left(\frac{3}{7}k-12\right)-9(\gamma k+8)=0\\2(\beta k-16)-6(\gamma k+8)=0\end{cases}$ | **M1** | 2.1 |
| | $\Rightarrow\beta k=\dfrac{2}{7}k+8$ and $\gamma k=\dfrac{1}{3}\left(\dfrac{2}{7}k-32\right)$<br>$\Rightarrow\dfrac{9k^2}{49}+\left(\dfrac{2}{7}k+8\right)^2+\dfrac{1}{9}\left(\dfrac{2}{7}k-32\right)^2=k^2\Rightarrow 2k^2-7k-490=0$ | **M1**<br>**A1** | 3.1a<br>1.1b |
| | $\Rightarrow(2k-35)(k+14)=0\Rightarrow k=\ldots$ | **M1** | 1.1b |

| | | | |
|---|---|---|---|
| | $k>0$ as direction cosine for first ordinate is positive, so need $k=\frac{35}{2}$<br>hence $\overrightarrow{OA}=\begin{pmatrix}\frac{3}{7}\times\frac{35}{2}\\ \frac{2}{7}\times\frac{35}{2}+8\\ \frac{1}{3}\left(\frac{2}{7}\times\frac{35}{2}-32\right)\end{pmatrix}=\ldots$ | **M1** | 2.3 |
| | Coordinates of $A$ are $\left(\frac{15}{2},13,-9\right)$ only | **A1** | 1.1b |
| | | **(7)** | |
| | | | **(7 marks)** |

**Way 1**

**B1:** Starts a correct procedure by parametrising the line correctly.

**M1:** Uses the direction cosine of $\frac{3}{7}$ to form an equation in $\lambda$

**M1:** Realises need to square, to form quadratic in $\lambda$ and gathers terms.

**A1:** Correct quadratic – three terms only or rearranged to complete square and solve, but need not have common factors all cancelled.

**M1:** Solves their three term quadratic, any valid method.

**M1:** Substitutes aa value for $\lambda$ into the equation of the line to find a position for $A$.

**A1:** Correct coordinates only

**Way 2**

**B1:** Starts correct procedure by using the direction cosines to parametrise $\overrightarrow{OA}$ or attempting to use the Pythagorean property of the direction cosines.

**M1:** Uses their $\overrightarrow{OA}$ as multiple of direction cosines in the line equation to produce simultaneous equations.

**M1:** Solves the system (no need to see check for consistency of third equation) to find $\beta k$ and $\gamma k$, or just $\beta$ and $\gamma$ in terms of $k$ and proceeds to form a quadratic in $k$ using the Pythagorean property of the direction cosines.

**A1:** A correct quadratic in $k$ reduced to three terms etc.

**M1:** Solves their three term quadratic, any valid method.

**M1:** Substitutes aa value for $\lambda$ into the equation of the line to find a position for $A$.

**A1:** Correct coordinates only