Number Theory

Questions

Q1.

The highest common factor of 963 and 657 is c.

(a) Use the Euclidean algorithm to find the value of c.

(3)

(b) Hence find integers a and b such that

$$963a + 657b = c$$

(3)

(Total for question = 6 marks)

Q2.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(i) (a) Use the Euclidean algorithm to find the highest common factor h of 416 and 72

(3)

(b) Hence determine integers a and b such that

$$416a + 72b = h$$

(3)

(c) Determine the value c in the set 0,1,2...,415 such that

$$23 \times 72 \equiv c \pmod{416}$$

(2)

(ii) Evaluate $5^{10} \pmod{13}$ giving your answer as the smallest positive integer solution.

(3)

(Total for question = 11 marks)

Q3.

(i) Use Fermat's Little Theorem to find the least positive residue of 6^{542} modulo 13

(5)

(ii) Seven students, Alan, Brenda, Charles, Devindra, Enid, Felix and Graham, are attending a concert and will sit in a particular row of 7 seats. Find the number of ways they can be seated if

(a) there are no restrictions where they sit in the row,

(1)

(b) Alan, Enid, Felix and Graham sit together,

(2)

(c) Brenda sits at one end of the row and Graham sits at the other end of the row,

(2)

(d) Charles and Devindra do not sit together.

(2)

(Total for question = 12 marks)

Q4.

(i) Determine all the possible integers a, where $a > 3$, such that

$$15 \equiv 3 \bmod a$$

(2)

(ii) Show that if p is prime, x is an integer and $x^2 \equiv 1 \bmod p$ then either

$$x \equiv 1 \bmod p \quad \text{or} \quad x \equiv -1 \bmod p$$

(3)

(iii) A company has £13 940 220 to share between 11 charities.

Without performing any division and showing all your working, decide if it is possible to share this money equally between the 11 charities.

(2)

(Total for question = 7 marks)

Q5.

In this question you must show detailed reasoning.

Without performing any division, explain why n = 20 210 520 is divisible by 66

(4)

(Total for question = 4 marks)

Q6.

(a) Use the Euclidean Algorithm to find integers a and b such that

$$125a + 87b = 1$$

(5)

(b) Hence write down a multiplicative inverse of 87 modulo 125

(1)

(c) Solve the linear congruence

$$87x \equiv 16 \pmod{125}$$

(2)

(Total for question = 8 marks)

Q7.

(a) Use the Euclidean algorithm to show that 124 and 17 are relatively prime (coprime).

(2)

(b) Hence solve the equation

$$124x + 17y = 10$$

(3)

(c) Solve the congruence equation

$$124x \equiv 6 \bmod 17$$

(2)

(Total for question = 7 marks)

Q8.

(i) Without performing any division, explain why 8184 is divisible by 6

(2)

(ii) Use the Euclidean algorithm to find integers a and b such that

$$27a + 31b = 1$$

(4)

(Total for question = 6 marks)

Q9.

(i) Using a suitable algorithm and without performing any division, determine whether 23 738 is divisible by 11

(2)

(ii) Use the Euclidean algorithm to find the highest common factor of 2322 and 654

(3)

(Total for question = 5 marks)

Q10.

(i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(3)

(ii) The digits which can be used in a security code are 1, 2, 3, 4, 5, 6, 7, 8 and 9

Originally the code used consisted of two distinct odd digits, followed by three distinct even digits.
To enable more codes to be generated, a new system is devised. This uses two distinct even digits, followed by any three other distinct digits. No digits are repeated.
Find the increase in the number of possible codes which results from using the new system.

(4)

(Total for question = 7 marks)

Mark Scheme – Number Theory

Q1.

Question	Scheme	Marks	AOs
(a)	$963 = 657\times1+306 \quad 657 = 306\times2+45$	M1	1.2
	$306 = 45\times6+36$ $45 = 36\times1+9$ $36 = 9\times4+0$	A1	1.1b
	$\text{HCF}(963,657) = 9$ or $c = 9$	A1	1.1b
		(3)	
(b)	$9 = 45-36\times1 \quad 9 = 45-(306-45\times6)$	M1	1.1b
	$9 = 7\times45-306$ $= 7\times(657-2\times306)-306 = 7\times657-15\times306$ $= 7\times657-15\times(963-1\times657)$	A1	2.1
	$9 = -15\times963+22\times657$	A1	1.1b
		(3)	
			(6 marks)

Notes

(a)
M1: Starts the process by showing 2 correct stages
A1: Completes the algorithm correctly
A1: Correct HCF
(b)
M1: Starts the reversal process by completing at least 2 stages
A1: Completes the algorithm correctly
A1: Correct values for a and b

Q2.

Question	Scheme	Marks	AOs
(i)(a)	$416 = 5\times 72 + 56$	M1	1.1b
	$72 = 1\times 56 + 16;\ 56 = 3\times 16 + 8; 16 = 2\times 8(+0)$	M1	1.1b
	Hence $h = 8$	A1	2.2a
		(3)	
(b)	Using back substitution $8 = 56 - 3\times 16$	M1	1.1b
	$= 56 - 3(72 - 1\times 56) = 4\times 56 - 3\times 72$ $= 4\times(416 - 5\times 72) - 3\times 72$	M1	1.1b
	$= 4\times 416 - 23\times 72$ (so $a = 4$ and $b = -23$)	A1	1.1b
		(3)	
(c)	$23\times 72 = 4\times 416 - 8 \equiv -8(\text{mod } 416)$ Or $23\times 72 = 1656 \Rightarrow 1656 - k416 = \ldots$	M1	3.1a
	$c = 416 - 8 = 408$	A1	1.1b
		(2)	
(ii)	E.g. $5^{10} \equiv (5^2)^5 \equiv 25^5 \equiv (-1)^5 \quad (\text{mod } 13)$	M1	1.1b
	$\equiv -1(\text{mod } 13)$	dM1	1.1b
	$\equiv 12(\text{mod } 13)$	A1	2.2a
		(3)	
			(11 marks)

Notes:
(i) (a) **M1:** Starts the process of using the algorithm, with attempt at $416 = p\times 72 + q$. **M1:** Continues the process until remainder zero is reached. **A1:** Deduces the correct highest common factor from correct work. **(b)** **M1:** Begins the process of back substitution by rearranging their equation with least positive remainder. **M1:** Completes the process. **A1:** Correct expression or values of a and b identified. **(c)** **M1:** Uses the answer to (b) and reduces modulo 416 to reach -8 as a congruent number. Or a full process to multiply $23\times 72 = 1656$ and reduce by subtracting multiples of 416 **A1:** For 408.
(ii) **M1:** Uses modulo 13 to reduce the equation in some way to reduce a power of 5, e.g. using $5^2 = 25 \equiv -1$ or 12. There will be lots of approaches that can be used here (e.g. $5^3 \equiv 125 \equiv -5(\text{mod } 13)$ is another possibility. Note the question instructs calculator solutions are not acceptable so finding 5^{10} itself is not acceptable.

dM1: Completes to a smallest (positive or negative) residue – allow if slips, but look for reducing to a number in $\{-6,\ldots 6\}$. Again other routes than the one shown are possible. E.g. $5^{10} \equiv (5^3)^3 \times 5 \equiv (-5)^3 \times 5 \equiv -125 \times 5 \equiv 5 \times 5 \equiv 25 \equiv 12 \pmod{13}$
A1: For 12.

Q3.

Question	Scheme	Marks	AOs
(i)	$6^{13-1} \equiv 1 \pmod{13}$ or $6^{13} \equiv 6 \pmod{13}$	B1	1.2
	Attempts $542 = 45 \times 12 + 2$ or $542 = 41 \times 13 + 9$ (seen or implied)	M1	1.1b
	$6^{542} = (6^{12})^{45} \times 6^2$ or $6^{542} = (6^{13})^{41} \times 6^9$	A1	1.1b
	$\equiv 1 \times 6^2 \equiv \ldots \pmod{13}$ or $\equiv 6^{41} \times 6^9 \equiv (6^{13})^3 \times 6^2 \times 6^9 \equiv 6^3 \times 6^2 \times 6^9 \equiv 6^{13} \times 6 \equiv 6^2 \equiv \ldots \pmod{13}$	M1	1.1b
	$\equiv 10 \pmod{13}$	A1	1.1b
		(5)	
(ii)(a)	$7! = 5040$	B1	1.1b
		(1)	
(b)	$4! \times 4! = 576$	M1 A1	3.1b 1.1b
		(2)	
(c)	$5! \times 2! = 240$	M1 A1	3.1b 1.1b
		(2)	
(d)	$7! - 6! \times 2! = 3600$ or $5! \times (2 \times 5 + 2 \times 4 + 2 \times 4 + 4) = 3600$	M1 A1	3.1b 1.1b
		(2)	
			(12 marks)

Notes
(i) B1: Recalls Fermat's Little Theorem correctly. May be implied in their work. M1: Attempts 542 in the form $12a + b$. Score if an attempt at $(6^{12})^{45}$ or similar is seen with attempt to combine with another term. Allow M1 for attempt "542 = their 12"$\times a + b$ A1: Uses their a and b to write 6^{542} correctly in terms of 6^{12} or 6^{13} (must be one of these two) M1: Completes the process to find the residue (more convoluted roots are possible but look for a complete process to reach a residue using their attempt at Fermat's Little Theorem at least once). A1: Correct residue. Allow if the "45" was incorrect so long as the remainder was 2. (ii)(a) B1: Correct value. Accept as 7! for this part but must be evaluated in the remaining parts. (b) M1: Evidence that the 4 students have been considered as one unit among many and sees the problem as permutations of 4 items. Score for $4!\times k$ where $k \neq 1$ A1: Correct value (c) M1: Realises that the other 5 students can sit in any position – evidenced by sight of 5! (in (c)) A1: Correct value (d) M1: A correct strategy applied. E.g. interprets the situation as the answer to part (ii)(a) minus the ways that they can sit together So score for $7! - ....$ or $5040 - ...$ where … is not zero. Alternatively, considers the different positions Devindra can sit for each different position for Charles, and with 5! positions for the rest. Look for 5!×(sum of 7 terms) oe. A1: Correct value

Q4.

Question	Scheme	Marks	AOs
(i)	For any correct value for $a = 4$, 6 or 12	M1	1.1b
	For all three correct values for a and no extras $a = 4$, 6 & 12	A1	1.1b
		(2)	
(ii)	$x^2 - 1$ is divisible by p **OR** $x^2 - 1 \equiv 0 \bmod p$ **OR** $p / (x^2 - 1)$	B1	1.1b
	$\therefore (x-1)(x+1)$ is divisible by p and **since p is prime** either $(x-1)$ is divisible by p or $(x+1)$ is divisible by p **OR** $\therefore (x-1)(x+1) \equiv 0 \bmod p$ and **since p is prime** either $x-1 \equiv 0 \bmod p$ or $x+1 \equiv 0 \bmod p$ **OR** $\therefore p/(x-1)(x+1)$ and **since p is prime** either $p/(x-1)$ or $p/(x+1)$	M1	2.1
	$\therefore \quad x \equiv 1 \bmod p \quad$ or $\quad x \equiv -1 \bmod p$ *	A1*	1.1b
		(3)	
(iii)	For selecting and performing a divisibility test for dividing by 11 $1-3+9-4+0-2+2-0=3$ or $1-3+9-4+0-2+2-0+0-0=3$ 3 is not divisible by 11 or $11 \nmid 3$ — or — Sum odd = 12 Sum even = 9 Difference = 3 which is not 0 or divisible by 11	M1	1.1b
	Fully correct method with reason (must have correct sum ± 3) and conclusion £13 940 220 is not divisible by 11 Therefore, it is **not is it possible** to share this **money equally** between the 11 **charities**	A1	3.2a
		(2)	
			(7 marks)

Notes
(i) **M1:** For an understanding of mod notation and finding a correct value for $a = 4$, 6 or 12 **A1:** For all three correct values for a and no extras $a = 4$, 6 & 12
(ii) see scheme
(iii) **M1:** For applying a divisibility test for dividing by 11 to £13 940 220 or 139 402 2000p **A1:** Fully correct method and concludes not divisible by 11 and interprets conclusion in context

Q5.

Question	Scheme	Marks	AOs
	Identifies either 3 or 11 as a prime divisor of 66 and proceeds to apply the divisibility test for this prime number.	**M1**	3.1a
	Either $2-0+2-1+0-5+2-0=0=0\times11$ hence n is divisible by 11 Or $2+0+2+1+0+5+2+0=12=4\times3$ hence n is divisible by 3	**A1**	2.2a
	Both $2-0+2-1+0-5+2-0=0=0\times11$ hence n is divisible by 11 And $2+0+2+1+0+5+2+0=12=4\times3$ hence n is divisible by 3	**A1**	2.2a
	As also n is even, it is divisible by 2, and hence as divisible by 2, 3 and 11, is divisible by $2\times3\times11=66$	**A1**	2.4
		(4)	
		(4 marks)	

Notes:

M1: Identifies one of the odd prime factors of 66 and proceeds to check divisibility for it.

A1: Correct method and deduction for either divisibility by 3 or by 11

A1: Correct method and deduction for both divisibility by 3 and by 11

A1: Notes also divisibility by 2 and explains why divisibility by 66 follows. The explanation may have been given in a preamble "$66 = 2\times3\times11$ so divisible by 66 if divisible by 2, 3 and 11"

The must be a correct reason for divisibility by 2, ie "it is even" or "last digit is even". Do not accept "last digit is 0" with no reason given.

NB There is no divisibility test for 6, so attempting such will result in the loss of the last two A marks. (E.g. sum of digits being a multiple of 6 is an incorrect test, as for instance 15 satisfies 1+5 = 6 but is not a multiple of 6.)

Q6.

Question	Scheme	Marks	AOs
(a)	$125 = 87\times1+38$ $87 = 38\times2+11...$	M1	1.1b
	$38 = 11\times3+5$ $11 = 5\times2+1$	M1 A1	1.1b 1.1b
	$1 = 11-5\times2$ $= 11-(38-11\times3)\times2 = 11\times7-38\times2$ $= (87-38\times2)\times7-38\times2 = 87\times7-38\times16$	M1	2.1
	$1 = 87\times7-(125-87\times1)\times16 = -16\times125+23\times87$ (So $a=-16$ and $b=23$)	A1	1.1b
		(5)	
(b)	From (a) $23\times87 \equiv 1(\text{mod}125)$ so multiplicative inverse of 87 is 23.	B1ft	2.2a
		(1)	
(c)	$x \equiv 23\times16(\text{mod } 125)$	M1	1.1b
	$x \equiv 368 \equiv 118(\text{mod } 125)$	A1	1.1b
		(2)	
			(8 marks)

Notes:
(a) **M1:** Begins the process of applying the Euclidean algorithm, with attempt at the first two steps. Allow slips. **M1:** Completes the process to the stage shown – if errors have been made at least three steps should have been made in reaching their final line (ending +1) to score this mark. **A1:** Algorithm correctly carried out – as shown. **M1:** Starts the process of back substitution – at least two substitutions made. **A1:** Completes the process and finds the correct values for a and b.
(b) **B1ft:** Deduces correct multiplicative inverse. Accept 23 or anything congruent to 23 modulo 125 or follow through their b.
(c) **M1:** Multiplies 16 by their multiplicative inverse or any other full method to proceed to the solution, e.g. multiplying the identity found in (a) through by 16 and reducing modulo 125. **A1:** $x \equiv 118(\text{mod } 125)$. Accept 118 or anything congruent to 118 modulo 125 as long as it is part of a correct modulo statement. (Do not accept just 368 on its own.)

Q7.

Question	Scheme	Marks	AOs
(a)	$124 = 17 \times 7 + 5$ $17 = 3 \times 5 + 2$ $5 = 2 \times 2 + 1$ $\{2 = 2 \times 1\}$	M1	2.1
	Since the **gcd** is 1, 124 and 17 are **relatively prime (coprime)**	A1	2.4
		(2)	
(b)	$1 = 5 - 2 \times 2$ $1 = 5 - 2(17 - 3 \times 5) \Rightarrow 1 = 7 \times 5 - 2 \times 17$ $1 = 7(124 - 17 \times 7) - 2 \times 17$ $\Rightarrow 1 = 7 \times 124 - 51 \times 17$	M1 A1	2.1 1.1b
	$\Rightarrow 10 = 70 \times 124 - 510 \times 17$ $x = 70,\ y = -510$	A1	2.2a
	Alternative For example $124 = 17 \times 7 + 5$ $5 = 124 - 7 \times 17$ | $1 = 5 - 2 \times 2$ $1 = 5 - 2(17 - 3 \times 5) \Rightarrow 4$ $= 2 \times 17 - 6 \times 5$ $4 = 2 \times 17 - 6(124 - 17 \times 7)$ $\Rightarrow 4 = 44 \times 17 - 6 \times 124$	M1 A1	2.1 1.1b
	$\Rightarrow 10 = 2 \times 124 - 14 \times 17$ $x = 2,\ y = -14$ | $\Rightarrow 10 = 110 \times 17 - 15 \times 124$ $x = -15,\ y = 110$	A1	2.2a
		(3)	
(c)	A complete method using modulo arithmetic to achieve $x \equiv \ldots \bmod 17$ For example $7 \times 124x \equiv 7 \times 6 \bmod 17 \Rightarrow x \equiv \ldots \bmod 17$	M1	2.1
	$x \equiv 8 \bmod 17$ o.e.	A1	1.1b
		(2)	
			(7 marks)

Notes:
(a) **M1:** Complete attempt at the Euclidean algorithm to find the gcd (hcf) of 124 and 17, condone a numerical slip. **A1:** A fully correct application of the Euclidean algorithm and draws the conclusion that since the **gcd/hcf = 1** therefore 124 and 17 are **relatively prime (coprime).**
(b) **M1:** Attempts to find the Bezout's identity, condone sign slips **A1:** Correct Bezout's identity. **A1:** Deduces a set of correct values of x and y.
Alternatively **M1:** Uses the solution to (a) to find an identity of the form $a \times 124 + b \times 17 = c$ **A1:** Correct identity **A1:** Deduces a set of correct values of x and y.
(c) **M1:** A complete method to reach using modulo arithmetic to achieve $x \equiv \ldots \mod 17$ For example Multiplies through by their multiplicative inverse of 124 and reaches $x \equiv \ldots \mod 17$. Bezout's identity used in part (b) follow through on their multiple of 124 **A1:** $x \equiv 8 \mod 17$o.e.

Q8.

Question	Scheme	Marks	AOs
(i)	Adding digits $8 + 1 + 8 + 4 = 21$ which is divisible by 3 (or continues to add digits giving 2+1=3 which is divisible by 3) so concludes that 8184 is divisible by 3	M1	1.1b
	8184 is even, so is divisible by 2 and as divisible by both 3 and 2, so it is divisible by 6	A1	1.1b
		(2)	
(ii)	Starts Euclidean algorithm $31=27 \times 1 + 4$ and $27 = 4 \times 6 + 3$	M1	1.2
	$4 = 3 \times 1 + 1$ (so hcf = 1)	A1	1.1b
	So $1 = 4 - 3 \times 1 = 4 - (27 - 4 \times 6) \times 1 = 4 \times 7 - 27 \times 1$	M1	1.1b
	$(31 - 27 \times 1) \times 7 - 27 \times 1 = 31 \times 7 - 27 \times 8$ $a = -8$ and $b = 7$	A1cso	1.1b
		(4)	
			(6 marks)

Notes:

(i)
M1: Explains divisibility by 3 rule in context of this number by adding digits
A1: Explains divisibility by 2, giving last digit even as reason and makes conclusion that number is divisible by 6

(ii)
M1: Uses Euclidean algorithm showing two stages
A1: Completes the algorithm. Does not need to state that hcf = 1
M1: Starts reversal process, doing two stages and simplifying
A1cso: Correct completion, giving clear answer following complete solution

Q9.

Question	Scheme	Marks	AOs
(i)	$2-3+7-3+8=\ldots$ or $2+7+8-(3+3)=\ldots$	M1	1.1b
	$=11$ so 23 738 is divisible by 11	A1	1.1b
		(2)	
(ii)	$2322=3\times 654+360,\quad 654=1\times 360+294$	M1	1.2
	$360=1\times 294+66,\quad 294=4\times 66+30$		
	$66=2\times 30+6,\quad 30=5\times 6+0$	A1	1.1b
	So HCF(2 322, 654) = 6	A1	1.1b
		(3)	
			(5 marks)

Notes

(i)
M1: Executes the correct process by adding and subtracting alternating digits or equivalent
A1: Completes correctly with a correct conclusion
(ii)
M1: Uses the Euclidean algorithm showing two stages (Must be Euclidean algorithm not e.g. using prime factors)
A1: Completes the algorithm correctly
A1: All correct and concludes HCF is 6

Q10.

Question	Scheme	Marks	AOs
(i)	$602 = 3\times161+119$	M1	1.1b
	$161 = 119+42,\ 119 = 2\times42+35$	M1	1.1b
	$42 = 35+7,\ 35 = 5\times7,\ \text{hcf} = 7$	A1	1.1b
		(3)	
(ii)	Number of codes under old system = $5\times4\times4\times3\times2\ (=480)$	B1	3.1b
	Number of codes under new system = $4\times3\times7\times6\times5\ (=2520)$	B1	3.1b
	Subtracts first answer from second	M1	1.1b
	Increase in number of codes is 2040	A1	1.1b
		(4)	
			(7 marks)

Notes:

(i)

M1: Attempts Euclid's algorithm – (there may be an arithmetic slip finding 119)

M1: Uses Euclid's algorithm a further two times with 161 and "their 119" and then with "their 119" and "their 42"

A1: This should be accurate with all the steps shown

(ii)

B1: Correctly interprets the problem and uses the five odd digits and four even digits to form a correct product

B1: Interprets the new situation using the four even digits, then the seven digits which have not been used, to form a correct product

M1: Subtracts one answer from the other

A1: Correct answer