Complex Numbers (CP2)

Questions

Q1.

Solve the equation

$$z^3 + 32 + 32\mathrm{i}\sqrt{3} = 0$$

giving your answers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$

(6)

(Total for question = 6 marks)

Q2.

The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + \mathrm{i}S = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

(Total for question = 8 marks)

Q3.

(a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7 \sin \theta - 56 \sin^3\theta + 112 \sin^5\theta - 64 \sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

(Total for question = 10 marks)

Q4.

(a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \text{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

(Total for question = 8 marks)

Q5.

In an Argand diagram, the points *A*, *B* and *C* are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number 6 + 2i.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + iy$, where x and y are real and exact.

(6)

The points *D*, *E* and *F* are the midpoints of the sides of triangle *ABC*.

(b) Find the exact area of triangle *DEF*.

(3)

(Total for question = 9 marks)

Q6.

A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

(Total for question = 7 marks)

Q7.

(a) Find the four roots of the equation $z^4 = 8(\sqrt{3} + i)$ in the form $z = re^{i\theta}$

(5)

(b) Show these roots on an Argand diagram.

(2)

(Total for question = 7 marks)

Q8.

(a) Use de Moivre's theorem to show that

$$\sin^5 \theta \equiv a \sin 5\theta + b \sin 3\theta + c \sin \theta$$

where a, b and c are constants to be found.

(5)

(b) Hence show that $\int_0^{\frac{\pi}{3}} \sin^5 \theta \, d\theta = \frac{53}{480}$

(5)

(Total for question = 10 marks)

Q9.

(i) The point P is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that P represents the complex number 6 + 6i, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $re^{i\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2i| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

(2)

(b) Determine the exact area of R, giving your answer in simplest form.

(4)

(Total for question = 11 marks)

Q10.

(a) Express the complex number $w = 4\sqrt{3} - 4\text{i}$ in the form $r(\cos\theta\ \theta + \text{i}\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$

(4)

(b) Show, on a single Argand diagram,

(i) the point representing w

(ii) the locus of points defined by $\arg(z + 10\text{i}) = \frac{\pi}{3}$

(3)

(c) Hence determine the minimum distance of w from the locus $\arg(z + 10\text{i}) = \frac{\pi}{3}$

(3)

(Total for question = 10 marks)

Q11.

(i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

(Total for question = 6 marks)

Mark Scheme – Complex Numbers (CP2)

Q1.

	Scheme	Notes	Marks
	$z^3+32+32\mathrm{i}\sqrt{3}=0$		
	$\arg(z^3)=\frac{4\pi}{3}$ or $-\frac{2\pi}{3}$	M1: Uses tan to find arg z^3 $\arctan\sqrt{3}$, $\arctan\frac{1}{\sqrt{3}}$, $\frac{\pi}{3}$ or $\frac{\pi}{6}$ seen. Allow equivalent angles A1: Either of values shown	M1A1
	$\lvert z\rvert=r=4$	Correct r seen anywhere (eg only in answers)	B1
	$3\theta=\frac{4\pi}{3},\ -\frac{2\pi}{3},\ -\frac{8\pi}{3}$		
	$\theta=\frac{4\pi}{9},\ -\frac{2\pi}{9},\ -\frac{8\pi}{9}$	Divides by 3 to obtain at least 2 values of θ which differ by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.	M1
	$\theta=\frac{4\pi}{9},\ -\frac{2\pi}{9}$ or $\frac{16\pi}{9},\ -\frac{8\pi}{9}$ or $\frac{10\pi}{9}$	At least 2 correct (and distinct) values from list shown	A1
	$z=4\mathrm{e}^{\frac{4\pi}{9}\mathrm{i}},\ 4\mathrm{e}^{-\frac{2\pi}{9}\mathrm{i}},\ 4\mathrm{e}^{-\frac{8\pi}{9}\mathrm{i}}$ or $4\mathrm{e}^{\mathrm{i}\theta}$ where $\theta=\ldots$	A1: All correct and in either of the forms shown Ignore extra answers outside the range	A1 (6)
			Total 6

Q2.

Question	Scheme	Marks	AOs
(a) Way 1	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos 5\theta+i\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+i\sin 9\theta)+...\right)$	M1	1.1b
	$=e^{i\theta}+\frac{1}{2}e^{5i\theta}\left(+\frac{1}{4}e^{9i\theta}+...\right)$	A1	2.1
	$C+iS=\frac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$	M1	3.1a
	$=\frac{2e^{i\theta}}{2-e^{4i\theta}}$ *	A1*	1.1b
		(4)	
(a) Way 2	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos 5\theta+i\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+i\sin 9\theta)+...\right)$	M1	1.1b
	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos\theta+i\sin\theta)^5\left(+\frac{1}{4}(\cos\theta+i\sin\theta)^9+...\right)$	A1	2.1
	$C+iS=\frac{\cos\theta+i\sin\theta}{1-\frac{1}{2}(\cos\theta+i\sin\theta)^4}=\frac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$	M1	3.1a
	$=\frac{2e^{i\theta}}{2-e^{4i\theta}}$ *	A1*	1.1b
		(4)	
(b) Way 1	$\frac{2e^{i\theta}}{2-e^{4i\theta}}\times\frac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$	M1	3.1a
	$\frac{4e^{i\theta}-2e^{-3i\theta}}{4-2e^{-4i\theta}-2e^{4i\theta}+1}$	A1	1.1b
	$\frac{4\cos\theta+4i\sin\theta-2\cos 3\theta+2i\sin 3\theta}{5-2\cos 4\theta+2i\sin 4\theta-2\cos 4\theta-2i\sin 4\theta}$ **Dependent on the first M**	dM1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
		(4)	

Question	Scheme	Marks	AOs
(b) Way 2	$\frac{2e^{i\theta}}{2-e^{4i\theta}}=\frac{2(\cos\theta+i\sin\theta)}{2-(\cos 4\theta+i\sin 4\theta)}\times\frac{2-(\cos 4\theta-i\sin 4\theta)}{2-(\cos 4\theta-i\sin 4\theta)}$	M1	3.1a
	$\frac{4\cos\theta+4i\sin\theta-2\cos\theta\cos 4\theta-2\sin\theta\sin 4\theta+2i\sin 4\theta\cos\theta-2i\sin\theta\cos 4\theta}{4+\cos^2 4\theta+\sin^2 4\theta-4\cos 4\theta}$	A1	1.1b
	$\frac{4\cos\theta+4i\sin\theta-2\cos 3\theta+2i\sin 3\theta}{5-2\cos 4\theta+2i\sin 4\theta-2\cos 4\theta-2i\sin 4\theta}$ **Dependent on the first M**	dM1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
		(8 marks)	

Notes
(a) **Way 1** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to Euler form correctly (At least up to 5θ) M1: Recognises that $C + iS$ is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors **Way 2** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to power form correctly (At least up to 5θ) M1: Recognises that $C + iS$ is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors (b) **Way 1** M1: Multiplies numerator and denominator by $2 - e^{-4i\theta}$ A1: Correct fraction in terms of exponentials **d**M1: Converts back to trigonometric form A1*: Reaches the printed answer with no errors **Way 2** M1: Converts back to trigonometric form and realises the need to make the denominator real and multiplies numerator and denominator by the complex conjugate of the denominator which is **correct** for their fraction A1: Correct fraction in terms of trigonometric functions **d**M1: Uses the correct addition formula to obtain $\sin 3\theta$ in the numerator A1*: Reaches the printed answer with no errors

Q3.

Question	Scheme	Marks	AOs
(a)	$(\cos\theta+\mathrm{i}\sin\theta)^7=\cos^7\theta+\binom{7}{1}\cos^6\theta(\mathrm{i}\sin\theta)+\binom{7}{2}\cos^5\theta(\mathrm{i}\sin\theta)^2+\ldots$ Some simplification may be done at this stage e.g. $c^7+7c^6\mathrm{i}s-21c^5s^2-35c^4\mathrm{i}s^3+35c^3s^4+21c^2\mathrm{i}s^5-7cs^6-\mathrm{i}s^7$	M1	1.1b
	$\mathrm{i}\sin 7\theta={}^7C_1c^6\mathrm{i}s+{}^7C_3c^4\mathrm{i}^3s^3+{}^7C_5c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$ or $=7c^6\mathrm{i}s+35c^4\mathrm{i}^3s^3+21c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$	M1	2.1
	$\sin 7\theta=7c^6s-35c^4s^3+21c^2s^5-s^7$	A1	1.1b
	$=7(1-s^2)^3s-35(1-s^2)^2s^3+21(1-s^2)s^5-s^7$ $=7(1-3s^2+3s^4-s^6)s-35(1-2s^2+s^4)s^3+21(1-s^2)s^5-s^7$	M1	2.1
	$\{7s-21s^3+21s^5-7s^7-35s^3+70s^5-35s^7+21s^5-21s^7-s^7\}$ leading to $\sin 7\theta=7\sin\theta-56\sin^3\theta+112\sin^5\theta-64\sin^7\theta$ *	A1*	1.1b
		(5)	
(b)	$1+\sin 7\theta=0\Rightarrow\sin 7\theta=-1$	M1	3.1a
	$7\theta=-450,-90,270,630,\ldots$ or $7\theta=-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2},\ldots$	A1	1.1b
	$\theta=-\frac{450}{7},-\frac{90}{7},\frac{270}{7},\frac{630}{7},\ldots\Rightarrow\sin\theta=\ldots$ or $\theta=-\frac{5\pi}{14},-\frac{\pi}{14},\frac{3\pi}{14},\frac{7\pi}{14},\ldots\Rightarrow\sin\theta=\ldots$	M1	2.2a
	$x=\sin\theta=-0.901,\ -0.223,\ 0.623,\ 1$	A1 A1	1.1b 2.3
		(5)	
			(10 marks)

Notes
(a) M1: Attempts to expand $(\cos\theta + i\sin\theta)^7$ including a recognisable attempt at binomial coefficients Some simplification may be done at this stage. (May only see imaginary terms) M1: Identifies imaginary terms with $\sin 7\theta$ A1: Correct expression with coefficients evaluated and i's dealt with correctly M1: Replaces $\cos^2\theta$ with $1-\sin^2\theta$ and applies the expansions of $(1-\sin^2\theta)^2$ and $(1-\sin^2\theta)^3$ to their expression A1*: Reaches the printed answer with no errors and expansion of brackets seen. (b) M1: Makes the connection with part (a) and realises the need to solve $\sin 7\theta = -1$ A1: At least one correct value for 7θ M1: Divides by 7 and deduces that x values are found by finding at least one value for $\sin\theta$ A1: Awrt 2 correct values for x A1: Awrt all 4 x values correct and no extras

Q4.

Question	Scheme	Marks	AOs
(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	

(b)(i)	$1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+...$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{2-(\cos\theta-\mathrm{i}\sin\theta)}{2-(\cos\theta-\mathrm{i}\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+...\right\}=\frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+...\right\}=\frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+...=\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b

	Alternative $1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+...$	M1	3.1a

	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}e^{i\theta}}\times\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}}=\frac{4-2e^{-i\theta}}{5-2\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+...=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0\Rightarrow\cos\theta=2$	M1	3.1a
	As $(-1\leq)\cos\theta\leq 1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1\leq\cos\theta\leq 1$ therefore $\frac{1}{6}<\frac{3}{2(5-4\cos\theta)}<\frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z}=ki\Rightarrow z=1+\frac{i}{k}$	M1	3.1a
	mod z >1 contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
			(8 marks)

Notes:
(a) **B1:** See scheme
(b)(i) **M1:** Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. **M1:** Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into their answer to part (a) and rationalises the denominator. **M1:** Equates the imaginary terms. **M1:** Multiplies out the denominator and simplifies by using the identity $\cos^2\theta+\sin^2\theta=1$

A1*: cso. Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series. Alternative **M1**: Substitutes $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. **M1**: Substitutes $z=\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ into their answer to part (a) and rationalises the denominator. **M1**: Uses $\mathrm{e}^{-\mathrm{i}\theta}=\cos\theta-\mathrm{i}\sin\theta$ and $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$ to express in terms of $\sin\theta$ and $\cos\theta$ **M1**: Select the imaginary terms. **A1***: cso Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series.
(b)(ii) **M1**: Setting the real part of the series = 0 and rearranges to find $\cos\theta=\ldots$ **A1**: See scheme
Alternative 1 **M1**: Rearranges imaginary part so that $\cos\theta$ only appears once **A1**: Uses $-1\le\cos\theta\le1$ to show that the sum must always be positive so must contain a real part
Alternative 2 **M1**: Sets sum as purely imaginary and rearranges to make z the subject **A1**: Shows a contradiction and draws an appropriate conclusion

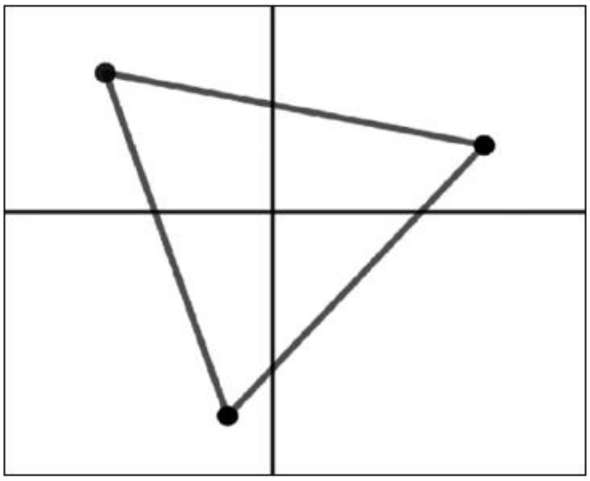
Q5.

Question	Scheme	Marks	AOs
(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
		(6)	
(b) Way 1	Area $ABC = 3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area $AOB=\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area $DEF=\frac{1}{4}ABC$ or $\frac{3}{4}AOB$	dM1	3.1a
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

(b) Way 2	$D\left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}\right)$ $OD = \sqrt{\left(\frac{3-\sqrt{3}}{2}\right)^2 + \left(\frac{3\sqrt{3}+1}{2}\right)^2} = \sqrt{10}$ Area $DOF = \frac{1}{2}\sqrt{10}\sqrt{10}\sin 120°$	M1	2.1
	Area $DEF = 3DOF$	**d**M1	3.1a
	$= 3 \times \frac{1}{2} \times \sqrt{10}\sqrt{10} \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 3	$AB = \sqrt{\left(9+\sqrt{3}\right)^2 + \left(3-3\sqrt{3}\right)^2} = \sqrt{120}$ Area $ABC = \frac{1}{2}\sqrt{120}\sqrt{120}\sin 60°\left(= 30\sqrt{3}\right)$	M1	2.1
	Area $DEF = \frac{1}{4}ABC$	**d**M1	3.1a
	$= \frac{1}{4} \times 30\sqrt{3} = \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 4	$D\left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}\right), E(-3,-1), F\left(\frac{3+\sqrt{3}}{2}, \frac{-3\sqrt{3}+1}{2}\right)$ $DE = \sqrt{\left(\frac{3-\sqrt{3}}{2}+3\right)^2 + \left(\frac{3\sqrt{3}+1}{2}+1\right)^2} \left(= \sqrt{30}\right)$ Area $DEF = \frac{1}{2}\sqrt{30}\sqrt{30}\sin 60°$	M1 **d**M1	2.1 3.1a
	$= \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 5	Area $ABC = \frac{1}{2}\begin{vmatrix} 6 & -3-\sqrt{3} & \sqrt{3}-3 & 6 \\ 2 & 3\sqrt{3}-1 & -3\sqrt{3}-1 & 2 \end{vmatrix} = 30\sqrt{3}$	M1	2.1
	Area $DEF = \frac{1}{4}ABC$	**d**M1	3.1a
	$= \frac{1}{4} \times 30\sqrt{3} = \frac{15\sqrt{3}}{2}$	A1	1.1b
			(9 marks)

Notes
(a) M1: Identifies a suitable method to rotate the given point by 120° (or equivalent) about the origin. May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply by $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ or $e^{\frac{2\pi}{3}i}$ A1: Correct real part or correct imaginary part A1: Completely correct complex number M1: Identifies a suitable method to rotate the given point by 240° (or equivalent e.g. rotate their *B* by 120°) about the origin May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply $6+2i$ by $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ or $e^{\frac{4\pi}{3}i}$ or their *B* by $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ or $e^{\frac{2\pi}{3}i}$ A1: Correct real part or correct imaginary part A1: Completely correct complex number (b) In general, the marks in (b) should be awarded as follows: M1: Attempts to find the area of a relevant triangle **d**M1: completes the problem by multiplying by an appropriate factor to find the area of *DEF* **Dependent on the first method mark** A1: Correct exact area In some cases it may not be possible to distinguish the 2 method marks. In such cases they can be awarded together for a direct method that finds the area of *DEF*

Examples:
Way 1 M1: A correct strategy for the area of a relevant triangle such as *ABC* or *AOB* **d**M1: Completes the problem by linking the area of *DEF* correctly with *ABC* or with *AOB* A1: Correct value **Way 2** M1: A correct strategy for the area of a relevant triangle such as *DOF* **d**M1: Completes the problem by linking the area of *DEF* correctly with *DOF* A1: Correct value **Way 3** M1: A correct strategy for the area of a relevant triangle such as *ABC* **d**M1: Completes the problem by linking the area of *DEF* correctly with *ABC* A1: Correct value **Way 4** M1**d**M1: A correct strategy for the area of *DEF*. Finds 2 midpoints and attempts one side of *DEF* and uses a correct triangle area formula. By implication this scores both M marks. A1: Correct value **Way 5** M1: A correct strategy for the area of *ABC* using the "shoelace" method. **d**M1: Completes the problem by linking the area of *DEF* correctly with *ABC* A1: Correct value **Note the marks in (b) can be scored using inexact answers from (a) and the A1 scored if an exact area is obtained.**



Q6.

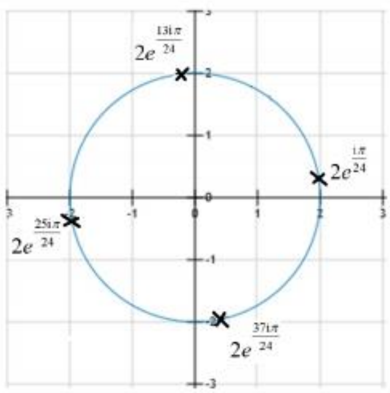
Question	Scheme	Marks	AOs
(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$*	A1*	1.1b
		(2)	
(b)	$(z+z^{-1})^4 = 16\cos^4\theta$	B1	2.1
	$(z+z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4(z^2 + z^{-2}) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	
			(7 marks)

Notes

(a)
M1: Identifies the correct form for z^n and z^{-n} and adds to progress to the printed answer
A1*: Achieves printed answer with no errors
(b)
B1: Begins the argument by using the correct index with the result from part (a)
M1: Realises the need to find the expansion of $(z+z^{-1})^4$
A1: Terms correctly combined
M1: Links the expansion with the result in part (a)
A1*: Achieves printed answer with no errors

Q7.

Question Number	Scheme	Notes	Marks
	$z^4 = 8(\sqrt{3}+\text{i})$		
(a)	$\left(\lvert z^4\rvert = \sqrt{(8\sqrt{3})^2 + 8^2} = \sqrt{256} =\right)$ 16 or $(\lvert z\rvert =)$ 2	Give B1 for either 16 or 2 seen anywhere	B1
	$(\arg z =)\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$	$\frac{\pi}{6}$ Accept 0.524	B1
	$r^4 = 16 \Rightarrow r = 2$		
	$4\theta = -\frac{23\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}$	Range not specified, you may see $4\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \frac{37\pi}{6}$	
	$\theta = -\frac{23\pi}{24}, -\frac{11\pi}{24}, \frac{\pi}{24}, \frac{13\pi}{24}$	Clear attempt at both r and θ with at least 2 different values for their $\arg z$, ie $r = \sqrt[4]{\text{their } 16}$, $\theta = \frac{\text{principal arg} + 2n\pi}{4}$ all 4 correct distinct values of θ cao. $\theta = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$ scores A1	M1, A1
	Roots are		
	$2\text{e}^{\frac{-23\text{i}\pi}{24}}, 2\text{e}^{\frac{-11\text{i}\pi}{24}}, 2\text{e}^{\frac{\text{i}\pi}{24}}, 2\text{e}^{\frac{13\text{i}\pi}{24}}$	All in correct form cao $2\text{e}^{\frac{\text{i}\pi}{24}}, 2\text{e}^{\frac{13\text{i}\pi}{24}}, 2\text{e}^{\frac{25\text{i}\pi}{24}}, 2\text{e}^{\frac{37\text{i}\pi}{24}}$ scores A1	A1
			(5)

(b)		B1: All 4 radius vectors to be the same length (approx) and perpendicular to each other. Circle not needed. Radius vector lines need not be drawn. If lines drawn and marked as perpendicular, accept for B1 B1: All in correct position relative to axes. Points marked must be close to the relevant axes. At least one point to be labelled or indication of scale given.	B1B1
			(2)
			Total 7
ALT:	Obtain one value - usually $2\text{e}^{\frac{\text{i}\pi}{24}}$ - and place on the circle. Position the other 3 by spacing evenly around the circle.		

Q8.

Question Number	Scheme	Notes	Marks
	$\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$		
(a)	$2\mathrm{i}\sin\theta = z - \frac{1}{z}$ or $2\mathrm{i}\sin n\theta = z^n - \frac{1}{z^n}$ oe	Seen anywhere "z" can be $\cos\theta + \mathrm{i}\sin\theta$ or $\mathrm{e}^{\mathrm{i}\theta}$ or z See below for use of $\mathrm{e}^{\mathrm{i}\theta}$	B1
	$\left(z-\frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right)$ $+10\left(z - \frac{1}{z}\right)$	M1: Attempt to expand powers of $z \pm \frac{1}{z}$ A1: Correct expression oe. A single power of z in each term. No need to pair. Must be numerical values; nCr s eg 5C2 score A0	M1A1
	$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$	At least one term on RHS correct – no need to simplify.	M1
	$= \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	All terms correct oe Decimals must be exact equivalents. a, b, c need not be shown explicitly. Must be in this form.	A1cso (5)
Use of $\mathrm{e}^{\mathrm{i}\theta}$	$2\mathrm{i}\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)$ oe		B1
	$(2i\sin\theta)^5 = \left(\left(\mathrm{e}^{5i\theta} - \mathrm{e}^{-5i\theta}\right) - 5\left(\mathrm{e}^{3i\theta} - \mathrm{e}^{-3i\theta}\right) + 10\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right)\right)$		M1A1
	$\left(32i\sin^5\theta =\right)\ \left(2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)\right)$ $\left(32\sin^5\theta =\right)\ \left(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta\right)$		M1
	$= \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$		A1cso

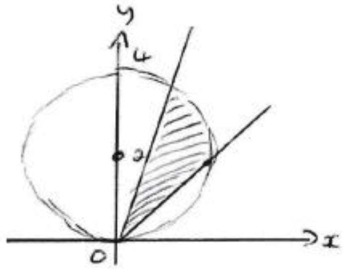
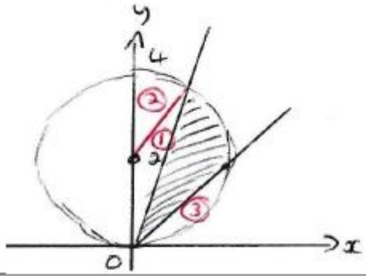
ALTs:			
Way 1	De Moivre on $\sin 5\theta$		
	$\sin 5\theta =$ $\mathrm{Im}(\cos 5\theta + \mathrm{i}\sin 5\theta) = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^5$	B1: $\sin 5\theta = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^5$	B1
	$= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$		
	$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta$ $+\sin^5\theta$	M1 Eliminate $\cos\theta$ from the expression using $\cos^2\theta = 1 - \sin^2\theta$ on at least one of the cos terms.	M1
	$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$	A1: Correct 3 term expression	A1
	Also: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$		
	Thus: $16\sin^5\theta = \sin 5\theta + 20\sin^3\theta - 5\sin\theta$		
	$= \sin 5\theta + 5(3\sin\theta - \sin 3\theta) - 5\sin\theta$	M1: Use their expression for $\sin 3\theta$ to eliminate $\sin^3\theta$	M1

	$=\sin 5\theta - 5\sin 3\theta + 10\sin\theta$		
	$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	A1:cso Correct result with no errors seen.	A1cso (5)
Way 2	De Moivre on $\sin 5\theta$ and use of compound angle formulae		
	$\sin 5\theta =$ $\text{Im}(\cos 5\theta + \text{i}\sin 5\theta) = \text{Im}(\cos\theta + \text{i}\sin\theta)^5$	B1: $\sin 5\theta = \text{Im}(\cos\theta + \text{i}\sin\theta)^5$	B1
	$=5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$		
	$=\frac{5}{2}\cos^3\theta\sin 2\theta - \frac{10}{4}\sin^2 2\theta\sin\theta + \sin^5\theta$	M1: Use $\sin 2\theta = 2\sin\theta\cos\theta$	M1
	$\sin^5\theta = \sin 5\theta - \frac{5}{4}(\sin 3\theta + \sin\theta)\cos^2\theta + \frac{10}{4}(1-\cos^2 2\theta)\sin\theta$		A1
	$=\sin 5\theta - \frac{5}{8}\cos\theta(\sin 4\theta + 2\sin 2\theta) + \frac{10}{4}\sin\theta - \frac{10}{8}(\sin 3\theta - \sin\theta)\cos^2\theta$		
	$=\sin 5\theta - \frac{5}{16}(\sin 5\theta + \sin 3\theta + 2(\sin 3\theta + \sin\theta))$ $+\frac{10}{4}\sin\theta - \frac{10}{16}(\sin 5\theta + \sin\theta - \sin 3\theta + \sin\theta)$		M1
	$=\frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	A1cso	A1cso
Way 3	Working from right to left:		
	$\sin 5\theta =$ $\text{Im}(\cos 5\theta + \text{i}\sin 5\theta) = \text{Im}(\cos\theta + \text{i}\sin\theta)^5$		B1
	$\sin 3\theta =$ $\text{Im}(\cos 3\theta + \text{i}\sin 3\theta) = \text{Im}(\cos\theta + \text{i}\sin\theta)^3$		
	$5a(1-2\sin^2\theta + \sin^4\theta)\sin\theta - 10a(1-\sin^2\theta)\sin^3\theta + a\sin^5\theta$ $+3b(1-\sin^2\theta)\sin\theta - b\sin^3\theta + c\sin\theta$ M1: Find the imaginary parts in terms of $\sin\theta$ and sub for $\sin 5\theta$, $\sin 3\theta$ in RHS A1: Correct (unsimplified) expression		M1A1
	$5a + 10a + a = 1$ $-10a - 10a - 3b - b = 0$ $5a + 3b + c = 0$	M1: Compare coefficients to obtain at least one of the equations shown	M1
	$a = \frac{1}{16},\ b = -\frac{5}{16},\ c = \frac{5}{8}$	A1cso	A1cso

(b)	$\int_0^{\frac{\pi}{3}} \sin^5\theta \, d\theta$ $= \frac{1}{32}\left[-\frac{2}{5}\cos 5\theta + \frac{10}{3}\cos 3\theta - 20\cos\theta\right]_0^{\frac{\pi}{3}}$ **NB: Penultimate A mark has been moved up to here.**	M1: $\sin n\theta \to \pm\frac{1}{n}\cos n\theta$ for $n = 3$ or 5 A1ft: 2 terms correctly integrated A1ft: Third term integrated correctly.	M1A1ft A1ft
	$= \left(-\frac{1}{160} - \frac{5}{48} - \frac{5}{16}\right) - \left(-\frac{1}{80} + \frac{5}{48} - \frac{5}{8}\right)$ $= -\frac{203}{480} - \left(-\frac{256}{480}\right)$	M1:Substitute both limits in a changed function to give numerical values. Incorrect integration such as $\pm n\cos n\theta$ could get M0A0A0M1A0	M1
	$\int_0^{\frac{\pi}{3}} \sin^5\theta = \frac{53}{480}$ **	cso, no errors seen.	A1cso (5) **Total 10**
OR:(b)	$\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$	Or their $a,b.c$ letters used or random numbers chosen	
	$\int_0^{\frac{\pi}{3}} \sin^5\theta \, d\theta = \left[-\frac{a}{5}\cos 5\theta - \frac{b}{3}\cos 3\theta - c\cos\theta\right]_0^{\frac{\pi}{3}}$	M1: $\sin n\theta \to \pm\frac{1}{n}\cos n\theta$ for $n = 3$ or 5 A1ft: Correct integration of their expression oe	
		M1:Substitute both limits, no trig functions	
		A0 A0 (A1s impossible here)	

Q9.

Question	Scheme	Marks	AOs
(i)	$\|z\| = \sqrt{6^2 + 6^2} = \ldots\ 6\sqrt{2}$ or $\sqrt{72}$ and $\arg z = \tan^{-1}\left(\frac{6}{6}\right) = \ldots\left\{\frac{\pi}{4}\right\}$ Can be implied by $r = 6\sqrt{2}e^{\frac{\pi}{4}i}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{5}$ to their argument $z = 6\sqrt{2}e^{\frac{\pi}{4}i} \times e^{\frac{2\pi k}{5}i}$ or $z = 6\sqrt{2}\left[\cos\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right)\right]$	M1	1.1b
	$z = re^{\left(\theta + \frac{2\pi}{5}\right)i}, re^{\left(\theta + \frac{4\pi}{5}\right)i}, re^{\left(\theta + \frac{6\pi}{5}\right)i}, re^{\left(\theta + \frac{8\pi}{5}\right)i}$ o.e. or $z = re^{\left(\theta + \frac{2\pi}{5}\right)i}, re^{\left(\theta - \frac{2\pi}{5}\right)i}, re^{\left(\theta - \frac{6\pi}{5}\right)i}, re^{\left(\theta - \frac{8\pi}{5}\right)i}$ o.e.	A1ft	1.1b
	$z = 6\sqrt{2}e^{\frac{13\pi}{20}i}, 6\sqrt{2}e^{\frac{21\pi}{20}i}, 6\sqrt{2}e^{\frac{29\pi}{20}i}, 6\sqrt{2}e^{\frac{37\pi}{20}i}$ o.e. or $z = 6\sqrt{2}e^{\frac{13\pi}{20}i}, 6\sqrt{2}e^{-\frac{19\pi}{20}i}, 6\sqrt{2}e^{-\frac{11\pi}{20}i}, 6\sqrt{2}e^{-\frac{3\pi}{20}i}$ o.e.	A1	1.1b
		(5)	

(ii)(a)	Circle centre (0, 2) and radius 2 or  with the point on the origin	B1	1.1b
	Fully correct 	B1	1.1b
		(2)	
(ii)(b)	area $=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 4\sin\theta^{\;2}\ d\theta$ or area $=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \alpha\sin\theta^{\;2}\ d\theta$	M1	3.1a
	Uses $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta+B\sin 2\theta$ area $=8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin^2\theta\ d\theta=4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}1-\cos 2\theta\ d\theta=4\theta-2\sin 2\theta$	M1	3.1a
	Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around $\left[4\left(\frac{\pi}{3}\right)-2\sin\left(\frac{2\pi}{3}\right)\right]-\left[4\left(\frac{\pi}{4}\right)-2\sin\left(\frac{2\pi}{4}\right)\right]$	M1	1.1b
	Area = $\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b
		(4)	
	Alternative 		
	Finds either the areas 1 or 2 Area 1 $=\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\left\{=\sqrt{3}\right\}$ Area 2 $=\frac{1}{2}\times 2^2\times\frac{\pi}{3}\left\{=\frac{2\pi}{3}\right\}$	M1	1.1b
	A complete method to find area 3 Area 3 $=\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\left\{=\pi-2\right\}$	M1	3.1a

	A complete method to find the required area Shaded area $=$ Area of semi circle $-$ area 1 $-$ area 2 $-$ area 3 $=\left[\frac{1}{2}\pi\times 2^2\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\frac{\pi}{3}\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=2\pi-\sqrt{3}-\frac{2\pi}{3}-(\pi-2)$ Or Shaded area $=$ Area of sector $-$ area 1 $-$ area 3 $=\left[\frac{1}{2}\times 4\times\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=\frac{4\pi}{3}-\sqrt{3}-(\pi-2)$	M1	3.1a
	Area $=\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b
		(4)	
			(11 marks)

Notes:

(i)
M1: Finds the modulus and argument of z
A1: Correct modulus and argument of z

M1: Uses a correct method to find to all the other 4 vertices of the pentagon. Must be doing the equivalent of adding/ subtracting multiples of $\frac{2\pi}{5}$ to the argument.

A1ft: All 4 vertices following through on their modulus and argument. Does not need to be simplified for this mark.
A1: All 4 vertices correct in the required form

(ii)(a)
B1: Circle centre (0, 2) and radius 2 or  with the vertex on the origin.
B1: Fully correct region shaded.

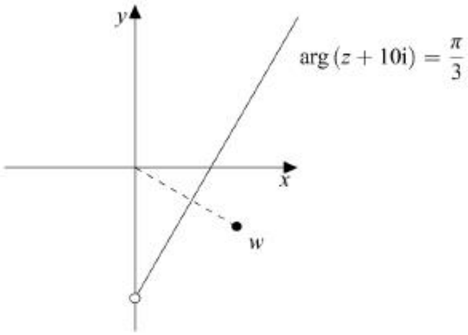
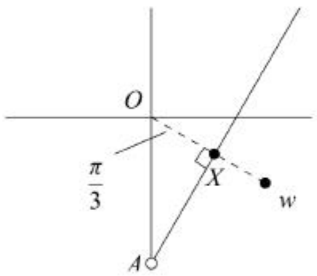
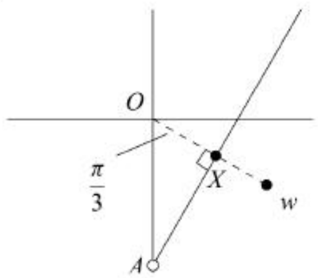
(ii) (b)
M1: Writes the required area using polar coordinates
M1: Uses $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta+B\sin 2\theta$

M1: Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around. Must be some attempt at area $=\frac{1}{2}\int \alpha\sin\theta^{\ 2}\ \mathrm{d}\theta$ and integration.

A1: Correct exact area $=\frac{\pi}{3}-\sqrt{3}+2$

Alternative
M1: Finds either area 1 or area 2
M1: A complete method to find the area 3
M1: A complete method to find the required area $=$ Area of semi circle $-$ area 1 $-$ area 2 $-$ area 3 or $=$ Area of sector $-$ area 1 $-$ area 3
A1: Correct exact area $=\frac{\pi}{3}-\sqrt{3}+2$

Q10.

Question	Scheme	Marks	AOs
(a)	$\lvert w\rvert=\sqrt{(4\sqrt{3})^2+(-4)^2}=8$	B1	1.1b
	$\arg w=\arctan\left(\frac{\pm 4}{4\sqrt{3}}\right)=\arctan\left(\pm\frac{1}{\sqrt{3}}\right)$	M1	1.1b
	$=-\frac{\pi}{6}$	A1	1.1b
	So $(w=)8\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$	A1	1.1b
		(4)	
(b)	 (i) w in 4th quadrant with either $(4\sqrt{3},-4)$ seen or $-\frac{\pi}{4}<\arg w<0$	B1	1.1b
	(ii) half line with positive gradient emanating from imaginary axis.	M1	1.1b
	The half line should pass between O and w starting from a point on the imaginary axis below w	A1	1.1b
		(3)	
(c)	 ΔOAX is right angled at X so $OX=10\sin\frac{\pi}{6}=5$ (oe)	M1	3.1a
	So shortest distance is $WX=OW-OX=$ '8' $-5=\ldots$	M1	1.1b
	So min distance is 3	A1	1.1b
	Alternative 1  A complete method to find the coordinates of X. Finds the equation of the line from O to w, $y=-\frac{1}{\sqrt{3}}x$ and the equation of the half line $y=\sqrt{3}x-10$, solves to find the point of intersection $X\left(\frac{5\sqrt{3}}{2},-\frac{5}{2}\right)$	M1	3.1a
	Finds the length WX $\sqrt{\left(4\sqrt{3}-\frac{5\sqrt{3}}{2}\right)^2+\left(-4--\frac{5}{2}\right)^2}$	M1	1.1b
	So min distance is 3	A1	1.1b
	Alternative 2	M1	3.1a

	Finds the length $AW = \sqrt{(4\sqrt{3}-0)^2+(-4--10)^2} = ...\{\sqrt{84}\}$ Finds the angle between the horizontal and the line AW $= \tan^{-1}\left(\frac{-4--10}{4\sqrt{3}}\right) = ...\{0.7137...\text{radians or } 40.89...^\circ\}$		
	Finds the length of $WX = \sqrt{84}\times\sin\left(\frac{\pi}{3}-0.7137\right) = ...$ Or $= \sqrt{84}\times\sin(60-40.89) = ...$	M1	1.1b
	So min distance is 3	A1	1.1b

	Alternative 3 Vector equation of the half line $r = \begin{pmatrix}0\\-10\end{pmatrix}+\lambda\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$ $XW = \begin{pmatrix}4\sqrt{3}-\lambda\\-4-\lambda\sqrt{3}-(-10)\end{pmatrix}$ Then either $\begin{pmatrix}4\sqrt{3}-\lambda\\6-\lambda\sqrt{3}\end{pmatrix}\bullet\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = 4\sqrt{3}-\lambda+6\sqrt{3}-3\lambda = 0 \Rightarrow \lambda = ...\left\{\frac{5}{2}\sqrt{3}\right\}$ $r = \begin{pmatrix}0\\-10\end{pmatrix}+\frac{5}{2}\sqrt{3}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = ...$ Or $XW^2 = (4\sqrt{3}-\lambda)^2+(6-\lambda\sqrt{3})^2 = 48-8\lambda\sqrt{3}+\lambda^2+36-12\lambda\sqrt{3}+3\lambda^2$ $xw^2 = 84-20\lambda\sqrt{3}+4\lambda^2$ leading to $\frac{d(XW^2)}{d\lambda} = -20\sqrt{3}+8\lambda = 0 \Rightarrow \lambda = ...$	M1	3.1a
	Finds the length WX $\sqrt{\left(4\sqrt{3}-\frac{5\sqrt{3}}{2}\right)^2+\left(-4--\frac{5}{2}\right)^2}$ Or $XW = \sqrt{\left(4\sqrt{3}-'\frac{5}{2}\sqrt{3}'\right)^2+\left(6-'\frac{5}{2}\sqrt{3}'\sqrt{3}\right)^2}$	M1	1.1b
	So min distance is 3	A1	1.1b
		(3)	
			(10 marks)

Notes:

(a)

B1: Correct modulus

M1: Attempts the argument. Allow for $\arctan\left(\frac{\pm4}{\pm4\sqrt{3}}\right)$ or equivalents using the modulus (may be in wrong quadrant for this mark).

A1: Correct argument $-\frac{\pi}{6}$ (must be in fourth quadrant but accept $\frac{11\pi}{6}$ or other difference of 2π for this mark).

A1: Correct expression found for w, in the correct form, must have positive $r = 8$ and $\theta = -\frac{\pi}{6}$.

Note: using degrees B1 M1 A0 A0

(b)(i)&(ii)

B1: w plotted in correct quadrant with either the correct coordinate clearly seen or above the line $y = -x$

M1: Half line drawn starting on the imaginary axis away from O with positive gradient (need not be labelled)

A1: Sketch on **one diagram**– both previous marks must have been scored and the half line should pass between O and w starting from a point on the imaginary axis below w. (You may assume it starts at -10i unless otherwise stated by the candidate)

Note: If candidates draw the loci on separate diagrams the maximum they can score is B1 M1 A0

(c)

M1: Formulates a correct strategy to find the shortest distance, e.g. uses right angle OXA where X is where the lines meet and proceeds at least as far as OX.

M1: Full method to achieve the shortest distance, e.g. for $WX = OW - OX$.

A1: cao shortest distance is 3

Alternative 1:

M1: Uses a correct method to find the equation of the line from O to w, $y = -\frac{1}{\sqrt{3}}x$ and the equation of the half line $y = \sqrt{3}x - 10$, solves to find the point of intersection $X\left(\frac{5\sqrt{3}}{2}, -\frac{5}{2}\right)$

If the incorrect gradient(s) is used with no valid method seen this is M0

M1: Finds the length $WX = \sqrt{\left(\text{their}\frac{5\sqrt{3}}{2} - 4\sqrt{3}\right)^2 + \left(\text{their} -\frac{5}{2} - -4\right)^2} = \ldots$ condone a sign slip in the brackets.

A1: cao shortest distance is 3

Alternative 2:

M1: Uses a correct method to find the length AW and a correct method to find the angle between the horizontal and the line AW

M1: Finds the length of $WX = \text{their}\sqrt{84} \times \sin\left(\frac{\pi}{3} - \text{their } 0.7137\right) = \ldots$

A1: cao shortest distance is 3

Alternative 3

M1: Finds the vector equation of the half line, then XW.

Then either: Sets dot product XW and the line = 0 and solves for λ. Substitutes their λ into the equation of the half line to find the point of intersection.

Or finds the length of XW and differentiates, set = 0 and solve for λ

M1: Finds the length $WX = \sqrt{\left(\text{their}\frac{5\sqrt{3}}{2} - 4\sqrt{3}\right)^2 + \left(\text{their} -\frac{5}{2} - -4\right)^2} = \ldots$ condone a sign slip in the brackets.

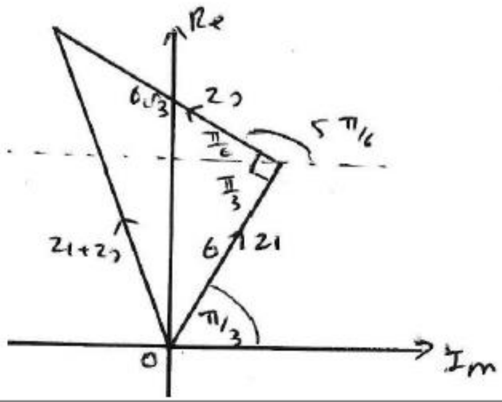
Or substitutes their value for λ into the length of *(d)*

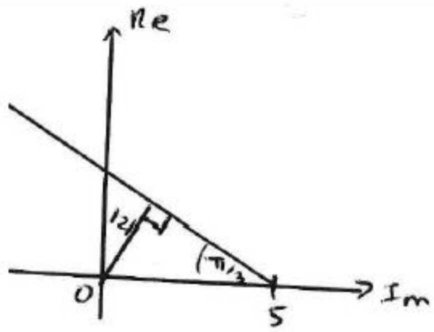
A1: cao shortest distance is 3

Q11.

Question	Scheme	Marks	AOs
(i)	$z_1 = 6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = \ldots\{3 + 3\sqrt{3}i\}$ $z_2 = 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = \ldots\{-9 + 3\sqrt{3}i\}$ $\{z_1 + z_2 =\}(3 + 3\sqrt{3}i) + (-9 + 3\sqrt{3}i) = \ldots\{-6 + 6\sqrt{3}i\}$ Or $\{z_1 + z_2 =\}6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] + 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = a + bi$ where a and b are constants, the trig function must be evaluated	M1	3.1a
	Clearly show the method to find modulus and argument for $z_1 + z_2$ $\arg(z_1 + z_2) = \pi - \tan^{-1}\left(\frac{6\sqrt{3}}{6}\right)$ or $\tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$ **and** $\lvert z_1 + z_2\rvert = \sqrt{6^2 + (6\sqrt{3})^2} = \ldots\{12\}$ **Alternative 1** $-6 + 6\sqrt{3}i = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 12\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ **Alternative 2** $12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= \ldots\{-6 + 6\sqrt{3}i\}$	dM1	2.1
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$ * $12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$ Therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*	A1*	1.1b
		(3)	

	Alternative 3 $z_1 + z_2 = 6e^{\frac{\pi}{3}i} + 6\sqrt{3}e^{\frac{5\pi}{6}i}$ $= 12\left[\frac{1}{2}\cos\left(\frac{\pi}{3}\right) + \frac{1}{2}i\sin\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{5\pi}{6}\right) + \frac{\sqrt{3}}{2}i\sin\left(\frac{5\pi}{6}\right)\right]$	M1	3.1a
	$12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 12\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$	dM1	2.1
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*	A1*	1.1b
		(3)	
	Alternative 4 $z_1 + z_2 = 6e^{\frac{\pi}{3}i} + 6\sqrt{3}e^{\frac{5\pi}{6}i} = 6e^{\frac{\pi}{3}i}\left(1 + \sqrt{3}e^{\frac{\pi}{2}i}\right) = 6e^{\frac{\pi}{3}i}(1 + \sqrt{3}i)$	M1	
	Either $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ **and** $\arg = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$	dM1	

	Or $6e^{\frac{\pi}{3}i}(1+\sqrt{3}i) = 12e^{\frac{\pi}{3}i}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i()e^{\frac{\pi}{3}i}\left(cos\left(\frac{\pi}{3}\right)+i\,sin\left(\frac{\pi}{3}\right)\right)\right)$		
	$z_1 + z_2 = 12e^{\frac{\pi}{3}i}e^{\frac{\pi}{3}i} = 12e^{\frac{2\pi}{3}i}*$	A1*	
		(3)	
	Alternative 5 Uses geometry to show that $z_1,\ z_2$ and z_1+z_2 form a right-angled triangle 	M1	3.1a
	$arg(z_1+z_2) = \frac{\pi}{3} + tan^{-1}\left(\frac{6\sqrt{3}}{6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$ $\lvert z_1+z_2\rvert = \sqrt{(6)^2+(6\sqrt{3})^2} = \ldots\{12\}$	dM1	1.1b
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}\ *$	A1*	1.1b
		(3)	

(ii)		M1	3.1a
	$sin\left(\frac{\pi}{3}\right) = \frac{\lvert z\rvert}{5} \Rightarrow \lvert z\rvert = \ldots$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	

	Alternative 1 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ or $\tan\left(\frac{\pi}{3}\right) = \frac{y}{5-x}$ $\|z\|^2 = x^2 + y^2 = x^2 + (-\sqrt{3}x + 5\sqrt{3})^2 = 4x^2 - 30x + 75$ $\frac{d\|z\|^2}{dx} = 8x - 30 = 0 \Rightarrow x = \ldots\{3.75\}$ or $\|z\|^2 = 4(x - 3.75)^2 + 18.75 \Rightarrow x = \ldots\ldots\{3.75\}$	M1	3.1a
	$\|z\| = \sqrt{4(\text{their}3.75)^2 - 30(\text{their}3.75) + 75}$	M1	1.1b
	$\|z\| = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
	Alternative 2 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ Perpendicular line through the origin $y = \frac{1}{\sqrt{3}}x$ and find the point of intersection of the two lines $\left(\frac{15}{4}, \frac{5\sqrt{3}}{4}\right)$	M1	3.1a
	Finds the distance from the origin to their point of intersection $\|z\| = \sqrt{\left(\text{their } \frac{15}{4}\right)^2 + \left(\text{their } \frac{5\sqrt{3}}{4}\right)^2} = \ldots$	M1	1.1b
	$\|z\| = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
			(6 marks)

Notes:

(i)

M1: A complete method to find both $z_1$and z_2 in the form $a + bi$and adds them together.

dM1: Dependent on previous method mark, finds the modulus and argument of $z_1 + z_2$. They must show their method, just stating modulus = 12 and argument = $\frac{2\pi}{3}$ is not sufficient as this is a show question.

Alternative 1: Factorises out 12 and find the argument

Alternative 2: uses $12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \ldots$

A1*: Achieves the correct answer following no errors or omissions.

Alternatively shows that $12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$and concludes therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$ *

Alternative 3

M1: Factorises out 12 and writes in the form
$12\left[\ldots\cos\left(\frac{\pi}{3}\right)+\ldots i\sin\left(\frac{\pi}{3}\right)+\ldots\cos\left(\frac{5\pi}{6}\right)+\ldots i\sin\left(\frac{5\pi}{6}\right)\right]$
dM1: Dependent on previous mark. Writes in the form $12(a+bi)$ leading to the form $12(\cos\theta + i\sin\theta)$
A1*: Achieves the correct answer following no errors or omissions.

<u>**Alternative 4**</u>
M1: Factorises out 6 and writes in the form $6e^{\frac{\pi}{3}i}\left(1+\sqrt{3}e^{\frac{\pi}{2}i}\right) = 6e^{\frac{\pi}{3}i}(1+ai)$
dM1: Dependent on previous method mark, finds the modulus and argument of $(1+ai)$ or $12(a+bi)$ leading to the form $12(\cos\theta + i\sin\theta)$
A1*: Achieves the correct answer following no errors or omissions.

<u>**Alternative 5**</u>
M1: Draws a diagram to show that z_1, z_2 and z_1+z_2 form a right-angled triangle.
dM1: Dependent on previous method mark, finds the modulus and argument of z_1+z_2
A1*: Achieves the correct answer following no errors or omissions.

Note: Writing $arg(z_1+z_2) = \arctan\left(\frac{6\sqrt{3}}{-6}\right) = -\frac{\pi}{3}$ therefore $arg(z_1+z_2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ with no diagram or finding z_1+z_2 is **M0dM0A0**

(ii)
M1: Draws a diagram and recognises that the shortest distance will form a right-angled triangle.
M1: Uses trigonometry to find the shortest length.
A1: Correct exact value.

<u>**Alternative 1**</u>
M1: Finds the equation of the half-line by attempting $m = -\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$. Finds x^2+y^2 in terms of x, differentiates, sets $= 0$ and finds the value of x.
M1: Uses their value of x to find the minimum value of $\sqrt{x^2+y^2}$
A1: Correct exact value.

<u>**Alternative 2**</u>
M1: Finds the equation of the half-line by attempting $m = -\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$. Finds the equation of the line perpendicular which passes through the origin. Finds the point of intersection of the lines
M1: Finds the distance from the origin to their point of intersection
A1: Correct exact value.