Complex Numbers

Questions

Q1.

(i) The complex number *w* is given by

$$w = \frac{p - 4\text{i}}{2 - 3\text{i}}$$

where p is a real constant.

(a) Express w in the form $a + b\text{i}$, where a and b are real constants.
Give your answer in its simplest form in terms of p.

(3)

Given that $\arg w = \frac{\pi}{4}$

(b) find the value of p.

(2)

(ii) The complex number *z* is given by

$$z = (1 - \lambda\text{i})(4 + 3\text{i})$$

where λ is a real constant.
Given that

$$|z| = 45$$

find the possible values of λ
Give your answers as exact values in their simplest form.

(3)

(Total for question = 8 marks)

Q2.

Given that 4 and $2i - 3$ are roots of the equation

$$x^3 + ax^2 + bx - 52 = 0$$

where a and b are real constants,

(a) write down the third root of the equation,

(1)

(b) find the value of a and the value of b.

(5)

(Total for question = 6 marks)

Q3.

(i) The complex number w is given by

$$w = \frac{p - 4\text{i}}{2 - 3\text{i}}$$

where p is a real constant.

(a) Express w in the form $a + b\text{i}$, where a and b are real constants.
Give your answer in its simplest form in terms of p.

(3)

Given that $\arg w = \frac{\pi}{4}$

(b) find the value of p.

(2)

(ii) The complex number z is given by

$$z = (1 - \lambda\text{i})(4 + 3\text{i})$$

where λ is a real constant.
Given that

$$|z| = 45$$

find the possible values of λ
Give your answers as exact values in their simplest form.

(3)

(Total for question = 8 marks)

Q4.

Given that 4 and 2i − 3 are roots of the equation

$$x^3 + ax^2 + bx - 52 = 0$$

where *a* and *b* are real constants,

(a) write down the third root of the equation,

(1)

(b) find the value of *a* and the value of *b*.

(5)

(Total for question = 6 marks)

Q5.

$$f(z) = z^4 - 6z^3 + pz^2 + qz + r$$

where p, q and r are real constants.

The roots of the equation $f(z) = 0$ are α, β, γ and δ where $\alpha = 3$ and $\beta = 2 + i$

Given that γ is a complex root of $f(z) = 0$

(a) (i) write down the root γ,

(ii) explain why δ must be real.

(2)

(b) Determine the value of δ.

(2)

(c) Hence determine the values of p, q and r.

(3)

(d) Write down the roots of the equation $f(-2z) = 0$

(2)

(Total for question = 9 marks)

Q6.

Given that $z = a + b\text{i}$ is a complex number where a and b are real constants,

(a) show that zz^* is a real number.

(2)

Given that

- $zz^* = 18$
- $\dfrac{z}{z^*} = \dfrac{7}{9} + \dfrac{4\sqrt{2}}{9}\text{i}$

(b) determine the possible complex numbers z

(5)

(Total for question = 7 marks)

Q7.

Let

$$f(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.
Given that the equation $f(z) = 0$ has distinct roots

$$\alpha,\ \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

(6)

(b) Hence find the value of p.

(2)

(Total for question = 8 marks)

Q8.

Let

$$f(z) = z^3 + pz^2 + qz - 15$$

where p and q are real constants.
Given that the equation $f(z) = 0$ has roots

$$\alpha, \frac{5}{\alpha} \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right)$$

(a) solve completely the equation $f(z) = 0$

(5)

(b) Hence find the value of p.

(2)

(Total for question = 7 marks)

Q9.

Let

$$z = \frac{4}{1+i}$$

Find, in the form $a + ib$ where $a, b \in \mathbb{R}$

(a) z

(2)

(b) z^2

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(c) find the value of p and the value of q.

(3)

(Total for question = 7 marks)

Q10.

$$z = \frac{4}{1+i}$$

Find, in the form $a + ib$ where a, $b \in \mathbb{R}$

(a) z

(2)

(b) z^2

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(c) find the value of p and the value of q.

(3)

(Total for question = 7 marks)

Mark Scheme – Complex Numbers

Q1.

Question Number	Scheme		Marks	
(i)		Mark (i)(a) and (i)(b) together.		
	$w = \dfrac{p-4\text{i}}{2-3\text{i}}$ $\quad \arg w = \dfrac{\pi}{4}$			
(a) **Way 1**	$w = \dfrac{(p-4\text{i})}{(2-3\text{i})} \times \dfrac{(2+3\text{i})}{(2+3\text{i})}$	Multiplies by $\dfrac{(2+3\text{i})}{(2+3\text{i})}$	M1	
	$= \left(\dfrac{2p+12}{13}\right) + \left(\dfrac{3p-8}{13}\right)\text{i}$	At least one of either the real or imaginary part of w is correct. Must be expanded but could be unsimplified e.g. expressed as single fraction. Condone $a+\text{i}b$.	A1	
		Correct w in its simplest form.	A1	
				[3]
(a) **Way 2**	$(a+\text{i}b)(2-3\text{i}) = (p-4\text{i})$			
	$2a+3b = p$	Multiplies out to obtain 2 equations in two unknowns.	M1	
	$3a-2b = 4$			
	$= \left(\dfrac{2p+12}{13}\right) + \left(\dfrac{3p-8}{13}\right)\text{i}$	At least one of either the real or imaginary part of w is correct. Must be expanded but could be unsimplified e.g. expressed as single fraction. Condone $a+\text{i}b$.	A1	
		Correct w in its simplest form.	A1	
				[3]
(b)	$\left\{\arg w = \dfrac{\pi}{4} \Rightarrow\right\}$ $\quad 2p+12 = 3p-8$ o.e. seen anywhere.	Sets the numerators of the real part of their w equal to the imaginary part of their w or if arctan used, require evidence of $\tan\dfrac{\pi}{4} = 1$	M1	
	$\Rightarrow p = 20$	$p = 20$	A1	
				[2]
(ii) **Way 1**	$z = (1-\lambda\text{i})(4+3\text{i})$ and $\lvert z\rvert = 45$			
	$\sqrt{1+\lambda^2}\,\sqrt{4^2+3^2}$	Attempts to apply $\lvert(1-\lambda\text{i})(4+3\text{i})\rvert = \sqrt{1+\lambda^2}\,\sqrt{4^2+3^2}$	M1	
	$\sqrt{1+\lambda^2}\,\sqrt{4^2+3^2} = 45$	Correct equation.	A1	
	$\{\lambda^2 = 9^2-1 \Rightarrow\}$ $\lambda = \pm 4\sqrt{5}$	$\lambda = \pm 4\sqrt{5}$	A1	
				[3]
Way 2	$z = (4+3\lambda)+(3-4\lambda)\text{i}$	Attempt to multiply out, group real and imaginary parts and apply the modulus.	M1	
	$\sqrt{(4+3\lambda)^2+(3-4\lambda)^2}$			
	$(4+3\lambda)^2+(3-4\lambda)^2 = 45^2$ or	Correct equation.	A1	
	$\sqrt{(4+3\lambda)^2+(3-4\lambda)^2} = 45$			
	$\{16+24\lambda+9\lambda^2+9-24\lambda+16\lambda^2 = 2025\}$	Condone if middle terms in expansions not explicitly stated.		
	$\{25\lambda^2 = 2000 \Rightarrow\}$ $\lambda = \pm 4\sqrt{5}$	$\lambda = \pm 4\sqrt{5}$	A1	
				[3] 8

	Question Notes
(ii)	**M1** Also allow $(1+\lambda^2)(4^2+3^2)$ for M1.
	M1 Also allow $(4+3\lambda)^2+(3-4\lambda)^2$ for M1.

Q2.

Question Number	Scheme		Marks
	$x^3+ax^2+bx-52=0$, $a,b\in\mathbb{R}$, 4 and $2i-3$ are roots		
(a)	$-2i-3$	$-2i-3$ seen anywhere in solution for Q6.	B1
			[1]
(b) Way 1	$(x-(2i-3))(x-"(-2i-3)")$; $=x^2+6x+13$ or $x=-3\pm 2i \Rightarrow (x+3)^2=-4$; $=x^2+6x+13(=0)$ $(x-4)(x-(2i-3))$; $=x^2-(1+2i)x+4(2i-3)$ $(x-4)(x-"(-2i-3)")$; $=x^2-(1-2i)x+4(-2i-3)$	Must follow from their part (a). Any incorrect signs for their part (a) in initial statement award M0; accept any equivalent expanded expression for A1.	M1; A1
	$(x-4)(x^2+6x+13)$ $\{=x^3+ax^2+bx-52\}$	$(x-3^{\text{rd}}$ root$)$(their quadratic) . Could be found by comparing coefficients from long division.	M1
	$a=2$, $b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 2	Sum $=(2i-3)+"(-2i-3)"=-6$ Product $=(2i-3)\times"(-2i-3)"=13$ So quadratic is $x^2+6x+13$	Attempts to apply either $x^2-(\text{sum roots})x+$ (product roots) $=0$ or $x^2-2\mathrm{Re}(\alpha)x+\lvert\alpha^2\rvert=0$	M1
		$x^2+6x+13$	A1
	$(x-4)(x^2+6x+13)$ $\{=x^3+ax^2+bx-52\}$	$(x-3^{\text{rd}}$ root$)$(their quadratic)	M1
	$a=2$, $b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]

(b) Way 3	$(2i-3)^3+a(2i-3)^2+b(2i-3)-52=0$ $5a-3b=43$ (real parts) and $6a-b=23$ (imaginary parts) or uses f(4) = 0 and f(a complex root) = 0 to form equations in a and b.	Substitutes $2i-3$ into the displayed equation and equates both real and imaginary parts.	M1
		$5a-3b=43$ and $6a-b=23$ **or** $16a+4b=-12$ and $(2i-3)^3+a(2i-3)^2+b(2i-3)-52=0$ / $(-2i-3)^3+a(-2i-3)^2+b(-2i-3)-52=0$	A1
	So $a=2, b=-11$ or $x^3+2x^2-11x-52$	Solves these equations simultaneously to find at least one of either $a=...$ or $b=...$	M1
		At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 4	$b=$ sum of product pairs	Attempts sum of product pairs.	M1
	$=4(2i-3)+4"(-2i-3)"+(2i-3)"(-2i-3)"$	All pairs correct o.e.	A1
	$a=-(\text{sum of 3 roots})=-(4+2i\ -3"-2i-3")$	Adds up all 3 roots	M1
	$a=2$, $b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 5	Uses f(4) = 0		M1
	$16a+4b=-12$		A1
	$a=-(\text{sum of 3 roots})=-(4+2i\ -3"-2i-3")$	Adds up all 3 roots	M1
	$a=2$, $b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5] 6

Q3.

Question Number	Scheme		Marks
(i)		Mark (i)(a) and (i)(b) together.	
	$w = \dfrac{p-4\mathrm{i}}{2-3\mathrm{i}}$ $\quad \arg w = \dfrac{\pi}{4}$		
(a) **Way 1**	$w = \dfrac{(p-4\mathrm{i})}{(2-3\mathrm{i})} \times \dfrac{(2+3\mathrm{i})}{(2+3\mathrm{i})}$	Multiplies by $\dfrac{(2+3\mathrm{i})}{(2+3\mathrm{i})}$	M1
	$= \left(\dfrac{2p+12}{13}\right) + \left(\dfrac{3p-8}{13}\right)\mathrm{i}$	At least one of either the real or imaginary part of w is correct. Must be expanded but could be unsimplified e.g. expressed as single fraction. Condone $a + \mathrm{i}b$.	A1
		Correct w in its simplest form.	A1
			[3]
(a) **Way 2**	$(a+\mathrm{i}b)(2-3\mathrm{i}) = (p-4\mathrm{i})$		
	$2a+3b=p$ $3a-2b=4$	Multiplies out to obtain 2 equations in two unknowns.	M1
	$= \left(\dfrac{2p+12}{13}\right) + \left(\dfrac{3p-8}{13}\right)\mathrm{i}$	At least one of either the real or imaginary part of w is correct. Must be expanded but could be unsimplified e.g. expressed as single fraction. Condone $a + \mathrm{i}b$.	A1
		Correct w in its simplest form.	A1
			[3]
(b)	$\left\{\arg w = \dfrac{\pi}{4} \Rightarrow\right\}$ $\quad 2p+12 = 3p-8$ o.e. seen anywhere.	Sets the numerators of the real part of their w equal to the imaginary part of their w or if arctan used, require evidence of $\tan\dfrac{\pi}{4} = 1$	M1
	$\Rightarrow p = 20$	$p = 20$	A1
			[2]
(ii) **Way 1**	$z = (1-\lambda\mathrm{i})(4+3\mathrm{i})$ and $\lvert z \rvert = 45$		
	$\sqrt{1+\lambda^2}\,\sqrt{4^2+3^2}$	Attempts to apply $\lvert(1-\lambda\mathrm{i})(4+3\mathrm{i})\rvert = \sqrt{1+\lambda^2}\,\sqrt{4^2+3^2}$	M1
	$\sqrt{1+\lambda^2}\,\sqrt{4^2+3^2} = 45$	Correct equation.	A1
	$\{\lambda^2 = 9^2 - 1 \Rightarrow\}$ $\lambda = \pm 4\sqrt{5}$	$\lambda = \pm 4\sqrt{5}$	A1
			[3]
Way 2	$z = (4+3\lambda) + (3-4\lambda)\mathrm{i}$ $\sqrt{(4+3\lambda)^2 + (3-4\lambda)^2}$	Attempt to multiply out, group real and imaginary parts and apply the modulus.	M1
	$(4+3\lambda)^2 + (3-4\lambda)^2 = 45^2$ or $\sqrt{(4+3\lambda)^2 + (3-4\lambda)^2} = 45$	Correct equation.	A1
	$\{16 + 24\lambda + 9\lambda^2 + 9 - 24\lambda + 16\lambda^2 = 2025\}$	Condone if middle terms in expansions not explicitly stated.	
	$\{25\lambda^2 = 2000 \Rightarrow\}$ $\lambda = \pm 4\sqrt{5}$	$\lambda = \pm 4\sqrt{5}$	A1
			[3] **8**

	Question Notes
(ii)	**M1** Also allow $(1+\lambda^2)(4^2+3^2)$ for M1.
	M1 Also allow $(4+3\lambda)^2 + (3-4\lambda)^2$ for M1.

Q4.

Question Number	Scheme		Marks
	$x^3+ax^2+bx-52=0$, $a,b\in\mathbb{R}$, 4 and $2\mathrm{i}-3$ are roots		
(a)	$-2\mathrm{i}-3$	$-2\mathrm{i}-3$ seen anywhere in solution for Q6.	B1
			[1]
(b) Way 1	$(x-(2\mathrm{i}-3))(x-"(-2\mathrm{i}-3)")$; $=x^2+6x+13$ or $x=-3\pm2\mathrm{i}\Rightarrow(x+3)^2=-4$; $=x^2+6x+13(=0)$ $(x-4)(x-(2\mathrm{i}-3))$; $=x^2-(1+2\mathrm{i})x+4(2\mathrm{i}-3)$ $(x-4)(x-"(-2\mathrm{i}-3)")$; $=x^2-(1-2\mathrm{i})x+4(-2\mathrm{i}-3)$	Must follow from their part (a). Any incorrect signs for their part (a) in initial statement award M0; accept any equivalent expanded expression for A1.	M1; A1
	$(x-4)(x^2+6x+13)$ $\{=x^3+ax^2+bx-52\}$	$(x-3^{\text{rd}}\text{ root})(\text{their quadratic})$. Could be found by comparing coefficients from long division.	M1
	$a=2,\ b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 2	Sum $=(2\mathrm{i}-3)+"(-2\mathrm{i}-3)"=-6$ Product $=(2\mathrm{i}-3)\times"(-2\mathrm{i}-3)"=13$ So quadratic is $x^2+6x+13$	Attempts to apply either $x^2-(\text{sum roots})x+(\text{product roots})=0$ or $x^2-2\mathrm{Re}(\alpha)x+\lvert\alpha^2\rvert=0$	M1
		$x^2+6x+13$	A1
	$(x-4)(x^2+6x+13)$ $\{=x^3+ax^2+bx-52\}$	$(x-3^{\text{rd}}\text{ root})(\text{their quadratic})$	M1
	$a=2,\ b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]

(b) Way 3	$(2\mathrm{i}-3)^3+a(2\mathrm{i}-3)^2+b(2\mathrm{i}-3)-52=0$ $5a-3b=43$ (real parts) and $6a-b=23$ (imaginary parts) or uses f(4) = 0 and f(a complex root) = 0 to form equations in a and b.	Substitutes $2\mathrm{i}-3$ into the displayed equation and equates both real and imaginary parts.	M1
		$5a-3b=43$ and $6a-b=23$ **or** $16a+4b=-12$ and $(2i-3)^3+a(2i-3)^2+b(2i-3)-52=0$ / $(-2i-3)^3+a(-2i-3)^2+b(-2i-3)-52=0$	A1
	So $a=2, b=-11$ or $x^3+2x^2-11x-52$	Solves these equations simultaneously to find at least one of either $a=\ldots$ or $b=\ldots$	M1
		At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 4	$b=$ sum of product pairs	Attempts sum of product pairs.	M1
	$=4(2\mathrm{i}-3)+4"(-2\mathrm{i}-3)"+(2\mathrm{i}-3)"(-2\mathrm{i}-3)"$	All pairs correct o.e.	A1
	$a=-(\text{sum of 3 roots})=-(4+2\mathrm{i}\ -3"-2\mathrm{i}-3")$	Adds up all 3 roots	M1
	$a=2,\ b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5]
(b) Way 5	Uses f(4) = 0		M1
	$16a+4b=-12$		A1
	$a=-(\text{sum of 3 roots})=-(4+2\mathrm{i}\ -3"-2\mathrm{i}-3")$	Adds up all 3 roots	M1
	$a=2,\ b=-11$ or $x^3+2x^2-11x-52$	At least one of $a=2$ or $b=-11$	A1
		Both $a=2$ and $b=-11$	A1
			[5] 6

Q5.

Question	Scheme	Marks	AOs
(a)(i)	$2-\mathrm{i}$	B1	1.2
(ii)	Roots of polynomials with real coefficients occur in conjugate pairs, β and γ form a conjugate pair, α is real so δ must also be real. or Quartics have either 4 real roots, 2 real roots and 2 complex roots or 4 complex roots. As 2 complex roots and 1 real root therefore so δ must also be real. or As α real and only one root δ remaining, if complex it would need to have a complex conjugate, which it can't have so must be real	B1	2.4
		(2)	
(b)	$\alpha+\beta+\gamma+\delta=6$ $\Rightarrow 3+2+\mathrm{i}+2-\mathrm{i}+\delta=6 \Rightarrow \delta=\ldots$	M1	3.1a
	$\delta=-1$	A1	1.1b
		(2)	
(c)	$\mathrm{f}(z)=(z-3)(z+1)(z-(2+\mathrm{i}))(z-(2-\mathrm{i}))=\ldots$ Alternative pair sum $=(3)(2+\mathrm{i})+(3)(2-\mathrm{i})+(3)(-1)+(-1)(2+\mathrm{i})$ $+(-1)(2-\mathrm{i})+(2+\mathrm{i})(2-\mathrm{i})=\ldots\{10\}$ triple sum $=(3)(2+\mathrm{i})(2-\mathrm{i})+(3)(-1)(2+\mathrm{i})$ $+(3)(-1)(2-\mathrm{i})+(-1)(2+\mathrm{i})(2-\mathrm{i})=\ldots\{-2\}$ product $=(3)(2+\mathrm{i})(2-\mathrm{i})(-1)=\ldots\{-15\}$	M1	3.1a
	$=(z^2-2z-3)(z^2-4z+5)$ $=z^4-6z^3+10z^2+2z-15$ $p=10, q=2, r=-15$	A1 A1	1.1b 1.1b
		(3)	
(d)	$z=\frac{1}{2},-\frac{3}{2}$	B1ft	1.1b
	$z=-1\pm\frac{\mathrm{i}}{2}$	B1ft	1.1b
		(2)	
			(9 marks)

Notes
(a)(i) B1: Correct complex number (a)(ii) B1: Correct explanation. (b) M1: Uses $2 \pm \mathrm{i}$ and 1 together with the sum of roots $= \pm\, 6$ to find a value for δ A1: Correct value (c) M1: Uses $(z-3)$ and $(z - \text{their } \delta)$ and their conjugate pair correctly as factors and makes an attempt to expand Alternatively attempts to find the pair sum, triple sum and product A1: Establishes at least 2 of the required coefficients correctly A1: Correct quartic or correct constants (d) B1ft: For $-\frac{3}{2}$ and $-\frac{\delta}{2}$ as the real roots B1ft: For $-1-\frac{\mathrm{i}}{2}$ and $-\frac{\gamma}{2}$ as the complex roots

Q6.

Question	Scheme	Marks	AOs
(a)	$z *= a - bi$then $zz *= (a+bi)(a-bi) = ...$	M1	1.1b
	$zz *= a^2 + b^2$ therefore, a real number	A1	2.4
		(2)	
(b)	$\frac{z}{z*} = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{(a^2-b^2)+2abi}{a^2+b^2} = \frac{7}{9} + \frac{4\sqrt{2}i}{9}$ or $\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{18} \Rightarrow$ $z^2 = 14 + 8\sqrt{2}i$ or $a + bi = \left(\frac{7}{9} + \frac{4\sqrt{2}i}{9}\right)(a - bi) = ..+..i$	M1	1.1b
	Forms two equations from $a^2 + b^2 = 18$ or $\frac{a^2-b^2}{18} = \frac{7}{9}$ or $\frac{a^2-b^2}{a^2+b^2} = \frac{7}{9}$ or $\frac{2ab}{18} = \frac{4\sqrt{2}}{9}$ or $\frac{2ab}{a^2+b^2} = \frac{4\sqrt{2}}{9}$ or $a = \frac{7}{9}a + \frac{4\sqrt{2}}{9}b$ oe	M1 A1	3.1a 1.1b
	Solves the equations simultaneously e.g. $a^2 + b^2 = 18$ and $a^2 - b^2 = 14$ leading to a value for a or b	dM1	1.1b
	$z = \pm(4 + \sqrt{2}i)$	A1	2.2a
		(5)	
			(7 marks)

Notes:

(a)(i)

M1: States or implies $z *= a - bi$and finds an expression for $zz *$

A1: Achieves $zz *= a^2 + b^2$and draws the conclusion that $zz *$is a real number. Accept $\in \mathbb{R}$ as conclusion, but not just "no imaginary part".

(b)

M1: Starts the process of solving by using the conjugate to form an equation with real denominators, and without z^* or i^2 in the equation. Accept as shown in scheme, or may multiply through by $a - bi$and expand and gather terms. May be implied by correct extraction of equation(s).

M1: Uses the given information to form two equations involving a and b at least one of which includes both. It must involve equating real or imaginary parts of $\frac{z}{z*} = \frac{7}{9} + \frac{4\sqrt{2}i}{9}$

A1: Any two correct equations arising from use of both given facts. (Note: if multiplying through by $a - bi$ then equating real and imaginary terms gives the same equation.)

dM1: Dependent on previous method mark, solves the equations to find a value for either a or b.

A1: Deduces the correct complex numbers and no extras. Do not accept $\pm 4 \pm \sqrt{2}i$

Note: it is possible to solve via polar coordinates, but unlikely to succeed. If you see responses you think are worthy of credit but are unsure how to mark, use review. Example solutions shown below.

(b) Alt	$\frac{z}{z^*}=\frac{z^2}{zz^*}=\frac{z^2}{18}\Rightarrow z^2=14+8\sqrt{2}i$ or let $\arg z=\theta$. then $\frac{z}{z*}=\frac{re^{i\theta}}{re^{-i\theta}}=e^{2i\theta}=\cos 2\theta+\mathrm{i}\sin 2\theta$	M1	1.1b
	$z^2=18(\cos\alpha+\mathrm{i}\sin\alpha)$where $\tan\alpha=\frac{4\sqrt{2}}{7}\Rightarrow z=\pm\sqrt{18}\left(\cos\frac{1}{2}\alpha+\mathrm{i}\sin\frac{1}{2}\alpha\right)$Or $\cos 2\theta+\mathrm{i}\sin 2\theta=\frac{7}{9}+\frac{4\sqrt{2}i}{9}\Rightarrow 2\cos^2\theta-1=\frac{7}{9}, 2\sin\theta\cos\theta=\frac{4\sqrt{2}}{9}$	M1 A1	1.1b 1.1b
	$\cos\frac{1}{2}\alpha=\sqrt{\frac{1}{2}(1+\cos\alpha)}=\sqrt{\frac{1}{2}\left(1+\frac{7}{9}\right)}=\ldots$ and $\sin\frac{1}{2}\alpha=\sqrt{\frac{1}{2}(1-\cos\alpha)}=\sqrt{\frac{1}{2}\left(1-\frac{7}{9}\right)}=\ldots$or $\Rightarrow\cos\theta=\frac{2\sqrt{2}}{3},\sin\theta=\frac{1}{3},r=\lvert z\rvert=\sqrt{zz^*}=\sqrt{18}$	dM1	3.1a
	$z=\pm(4+\sqrt{2}i)$	A1	2.2a
		(5)	

Q7.

Question	Scheme	Marks	AOs
(a)	$\alpha+\beta+\left(\alpha+\frac{12}{\alpha}-\beta\right)=8$ so $2\alpha+\frac{12}{\alpha}=8$	M1 A1	1.1b 1.1b
	$\Rightarrow 2\alpha^2-8\alpha+12=0$ or $\alpha^2-4\alpha+6=0$ $\Rightarrow\alpha=\frac{4\pm\sqrt{(-4)^2-4(1)(6)}}{2(1)}$ or $(\alpha-2)^2-4+6=0\Rightarrow\alpha=\ldots$	M1	1.1b
	$\Rightarrow\alpha=2\pm\mathrm{i}\sqrt{2}$ are the two complex roots	A1	1.1b
	A correct full method to find the third root. Common methods are: Sum of roots $=8\Rightarrow$ third root $=8-\left(2+\mathrm{i}\sqrt{2}\right)-\left(2-\mathrm{i}\sqrt{2}\right)=\ldots$ third root $=2+\mathrm{i}\sqrt{2}+\frac{12}{2+\mathrm{i}\sqrt{2}}-\left(2-\mathrm{i}\sqrt{2}\right)=\ldots$ Product of roots $=24\Rightarrow$ third root $=\frac{24}{\left(2+\mathrm{i}\sqrt{2}\right)\left(2-\mathrm{i}\sqrt{2}\right)}=\ldots$ $(z-\alpha)(z-\beta)=z^2-4z+6\Rightarrow\mathrm{f}(z)=\left(z^2-4z+6\right)(z-\gamma)\Rightarrow\gamma=\ldots$ (or long division to find third factor).	M1	3.1a
	Hence the roots of $\mathrm{f}(z)=0$ are $2\pm\mathrm{i}\sqrt{2}$ and 4	A1	1.1b
		(6)	
(b)	E.g. f(4) = 0 $\Rightarrow 4^3-8\times4^2+4p-24=0\Rightarrow p=\ldots$ Or $p=\left(2+\mathrm{i}\sqrt{2}\right)\left(2-\mathrm{i}\sqrt{2}\right)+4\left(2+\mathrm{i}\sqrt{2}\right)+4\left(2-\mathrm{i}\sqrt{2}\right)\Rightarrow p=\ldots$ Or $\mathrm{f}(z)=(z-4)(z^2-4z+6)\Rightarrow p=\ldots$	M1	3.1a
	$\Rightarrow p=22$ cso	A1	1.1b
		(2)	
		(8 marks)	

Notes		
(a)	**M1**	Equates sum of roots to 8 and obtains an equation in just α.
	A1	Obtains a correct equation in α.
	M1	Forms a three term quadratic equation in α and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha = ...$
	A1	$\alpha = 2 \pm \mathrm{i}\sqrt{2}$
	M1	Any correct method for finding the remaining root. There are various routes possible. See scheme for common ones. Allow this mark if -24 is used as the product. See note below for a less common approach.
	A1	Third root found with all three roots correct. Note α and β need not be identified.
(b)	**M1**	Any correct method of finding p. For example, applies the factor theorem, process of finding the pair sum of roots, or uses the roots to form f(z).
	A1	$p = 22$ by correct solution only. Note: this can be found using only their complex roots from (a) (e.g. by factor theorem)

Note for (a) final M – it is possible to find the second and third roots using only one initial root (e.g. if second root forgotten or error leads to only one initial root being found).

Product of roots = $\alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right) = 24 \Rightarrow \alpha\beta^2 - \left(\alpha^2 + 12\right)\beta + 24 = 0$, substitutes in α and attempts to solve the quadratic in β to achieve remaining roots. The final M can be gained once three roots in total have been obtained. (This is unlikely to be seen as part of a correct answer.) Allow if -24 has been used for the product.

Q8.

Question	Scheme	Marks	AOs
(a)	$\alpha\left(\frac{5}{\alpha}\right)\left(\alpha+\frac{5}{\alpha}-1\right)=15$	M1 A1	1.1b 1.1b
	$\Rightarrow 5\alpha+\frac{25}{\alpha}-5=15 \Rightarrow \alpha^2-4\alpha+5=0$ $\Rightarrow \alpha=\frac{--4\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$ or $(\alpha-2)^2-4+5=0 \Rightarrow \alpha=...$	M1	3.1a
	$\Rightarrow \alpha=2\pm\mathrm{i}$	A1	1.1b
	Hence the roots of $\mathrm{f}(z)=0$ are $2+\mathrm{i}$, $2-\mathrm{i}$ and 3	A1	2.2a
		(5)	
(b)	$p=-\left(\text{"}(2+\mathrm{i})\text{"}+\text{"}(2-\mathrm{i})\text{"}+\text{"}3\text{"}\right) \Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
(b) ALT 1	$\mathrm{f}(z)=(z-3)(z^2-4z+5) \Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
	(7 marks)		

	Question Notes	
(a)	M1	Multiplies the three given roots together and sets the result equal to 15 or -15
	A1	Obtains a correct equation in α.
	M1	Forms a quadratic equation in α and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha=....$
	A1	$\alpha=2\pm\mathrm{i}$
	A1	Deduces the roots are $2+\mathrm{i}$, $2-\mathrm{i}$ and 3
(b)	M1	Applies the process of finding $-\sum\left(\text{of their three roots found in part }(a)\right)$ to give $p=...$
	A1	$p=-7$ by correct solution only.
(b) ALT 1	M1	Applies the process expanding $(z-\text{"}3\text{"})(z-(\text{their sum})z+\text{their product})$ in order to find $p=...$
	A1	$p=-7$ by correct solution only.

Q9.

Question Number	Scheme	Marks
	(a) $z = \dfrac{4(1-\text{i})}{(1+\text{i})(1-\text{i})}$	M1
	$z = 2(1-\text{i})$ or $2 - 2\text{i}$ or exact equivalent.	A1 (2)
	(b) $z^2 = (2-2\text{i})(2-2\text{i}) = 4 - 8\text{i} + 4\text{i}^2$	M1
	$= -8\text{i}$	A1 cao (2)
	(c) If z is a root so is z^* So $(x-2+2\text{i})(x-2-2\text{i})$ (or $x^2 - 2\text{Re}(z).x + \lvert z\rvert^2$)	M1
	So $(x-2+2\text{i})(x-2-2\text{i}) = 0$ (or $x^2 - 2\text{Re}(z).x + \lvert z\rvert^2 = 0$) and so $p = q =$	M1
	Equation is $x^2 - 4x + 8 (= 0)$ or $p = -4$ and $q = 8$	A1 (3) (7 marks)
ALT 1	(c) Substitutes $z = 2 - 2\text{i}$ and $z^2 = -8\text{i}$ into quadratic and equates real and imaginary parts to obtain $2p + q = 0$ and $-2p - 8 = 0$ Attempts to solve simultaneous equations to obtain $p = -4$ and $q = 8$	M1 M1A1
ALT 2	(c) Attempts to obtain $p = -$ sum of roots Attempts product of roots to obtain $q=$	 M1 M1
	Equation is $x^2 - 4x + 8 (= 0)$ or $p = -4$ and $q = 8$	A1
ALT 3	(c) $x - 2 = \pm 2\text{i}$ either sign acceptable	M1
	$(x-2)^2 = -4 \Rightarrow x^2 - 4x + 4 = -4$ i.e square and attempt to expand to give 3-term quadratic	M1
	Equation is $x^2 - 4x + 8 (= 0)$ or $p = -4$ and $q = 8$	A1

Notes

(a) M1: Multiplies numerator and denominator by $1 - \text{i}$ or by $-1 + \text{i}$

A1: cao

(b) M1: Squares their z, or the given $z = \dfrac{4}{1+\text{i}}$, to produce at least 3 terms which can be implied by the correct answer.

A1: -8i or $0 - 8\text{i}$ only

(c) M1: Uses their z and z^* in $(x-z)(x-z^*)$

M1: Multiplies two factors and obtains $p =$ or $q =$

A1: Both correct required – can be implied by $x^2 - 4x + 8$

ALT 1

(c) M1: Substitutes their z and their z^2 into the quadratic and equates real and imaginary parts to obtain two equations in p and q

M1: Attempts to solve for one unknown to obtain $p =$ or $q =$

A1: Both correct required – can be implied by $x^2 - 4x + 8 (= 0)$

Q10.

Question Number	Scheme	Marks
	(a) $z = \frac{4(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$	M1
	$z = 2(1-\mathrm{i})$ or $2 - 2\mathrm{i}$ or exact equivalent.	A1 (2)
	(b) $z^2 = (2-2\mathrm{i})(2-2\mathrm{i}) = 4 - 8\mathrm{i} + 4\mathrm{i}^2$	M1
	$= -8\mathrm{i}$	A1 cao (2)
	(c) If z is a root so is z^* So $(x-2+2\mathrm{i})(x-2-2\mathrm{i})$ (or $x^2 - 2\mathrm{Re}(z).x + \lvert z\rvert^2$)	M1
	So $(x-2+2\mathrm{i})(x-2-2\mathrm{i}) = 0$ (or $x^2 - 2\mathrm{Re}(z).x + \lvert z\rvert^2 = 0$) and so $p = q =$	M1
	Equation is $x^2 - 4x + 8(=0)$ or $p = -4$ and $q = 8$	A1 (3) (7 marks)
ALT 1	(c) Substitutes $z = 2 - 2\mathrm{i}$ and $z^2 = -8\mathrm{i}$ into quadratic and equates real and imaginary parts to obtain $2p + q = 0$ and $-2p - 8 = 0$	M1
	Attempts to solve simultaneous equations to obtain $p = -4$ and $q = 8$	M1A1
ALT 2	(c) Attempts to obtain $p = -$ sum of roots Attempts product of roots to obtain $q=$	M1 M1
	Equation is $x^2 - 4x + 8(=0)$ or $p = -4$ and $q = 8$	A1
ALT 3	(c) $x - 2 = \pm 2\mathrm{i}$ either sign acceptable	M1
	$(x-2)^2 = -4 \Rightarrow x^2 - 4x + 4 = -4$ i.e square and attempt to expand to give 3-term quadratic	M1
	Equation is $x^2 - 4x + 8(=0)$ or $p = -4$ and $q = 8$	A1

Notes

(a) M1: Multiplies numerator and denominator by $1 - \mathrm{i}$ or by $-1 + \mathrm{i}$
A1: cao
(b) M1: Squares their z, or the given $z = \frac{4}{1+\mathrm{i}}$, to produce at least 3 terms which can be implied by the correct answer.
A1: $-8\mathrm{i}$ or $0 - 8\mathrm{i}$ only
(c) M1: Uses their z and z^* in $(x - z)(x - z^*)$
M1: Multiplies two factors and obtains $p =$ or $q =$
A1: Both correct required – can be implied by $x^2 - 4x + 8$
ALT 1
(c) M1: Substitutes their z and their z^2 into the quadratic and equates real and imaginary parts to obtain two equations in p and q
M1: Attempts to solve for one unknown to obtain $p =$ or $q =$
A1: Both correct required – can be implied by $x^2 - 4x + 8(=0)$