OCR
Oxford Cambridge and RSA

Friday 22 October 2021 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

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You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. You can use extra paper if you need to, but you must clearly show your candidate number, the centre number and the question numbers.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 A family of circles is given by the equation

$$(x-2\cos a)^2+(y-2\sin a)^2=1 \ (*)$$

where the parameter a satisfies $0 \leqslant a < 2\pi$.

(a) Use a slider (for a) to investigate this family of circles. Write down the cartesian equation of the curve which contains the centre of each circle in the family. **[1]**

(b) Let b and c be real numbers with $0 \leqslant b < c < \pi$. Find and simplify an expression, in terms of b and c, for the distance between the centre of the circle corresponding to $a = b$ and the centre of the circle corresponding to $a = c$. **[2]**

(c) Hence, or otherwise, find a condition on b and c for the two circles in part **(b)** to touch. **[2]**

A curve which every member of a family of curves or lines touches tangentially is called an *envelope* of the family.

(d) By tracing the family of curves using a slider (for a), or otherwise, sketch the envelope of the family (*) in the Printed Answer Booklet. **[2]**

(e) Write down the equations of the curves which make up the envelope for this family (*). **[2]**

2 This question is about the family of straight lines which pass through the points $(0, a)$ and $(1, a^2)$ where the parameter a is any real number.

(a) In terms of a, find the equation of the straight line which passes through the points $(0, a)$ and $(1, a^2)$. **[2]**

(b) Let b and c be distinct real numbers. Given that the straight line corresponding to $a = b$ and the straight line corresponding to $a = c$ are parallel, find b in terms of c. **[3]**

(c) By tracing the family using a slider (for a), or otherwise, sketch the envelope of this family in the Printed Answer Booklet. **[2]**

(d) Determine, in the form $y = \mathrm{h}(x)$, the cartesian equation of the envelope for this family. **[5]**

3 **(a)** **(i)** Create a program which returns the highest common factor of positive integers m and n. Write out your program in full in the Printed Answer Booklet. **[3]**

In the rest of this question the highest common factor of positive integers m and n is denoted by (m, n).

(ii) Use your program to find (74333, 89817). **[1]**

(b) Euler's totient function $\varphi(n)$, where n is a positive integer, is defined to be the number of integers m with $1 \leqslant m \leqslant n$ such that $(m, n) = 1$.
For example $\varphi(6) = 2$ because $(1, 6) = 1$, $(2, 6) = 2$, $(3, 6) = 3$, $(4, 6) = 2$, $(5, 6) = 1$ and $(6, 6) = 6$.

(i) Extend your program in **(a)(i)** to create a program which returns $\varphi(n)$ for a given positive integer n. **[3]**

(ii) Use your program to find $\varphi(128)$ and $\varphi(1000)$. **[2]**

(iii) For a positive integer n, determine $\varphi(2^n)$ in terms of n. **[2]**

(iv) For a positive integer n, determine $\varphi(10^n)$ in terms of n. **[3]**

(c) For any positive integer k, let $\mathrm{F}(k)$ be the number of distinct fractions $\frac{m}{n}$ where $0 < m < n \leqslant k$.
For example F(4) = 5, since there are five fractions which satisfy the required condition, namely $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

(i) Find $\mathrm{F}(5)$ and $\mathrm{F}(6)$. **[2]**

(ii) Explain why, for any positive integer l, $\mathrm{F}(l+1) = \mathrm{F}(l) + \varphi(l+1)$. **[2]**

(iii) Determine $\mathrm{F}(100)$. **[2]**

4 This question concerns the family of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2(x+1)} + a\arctan(y) \quad (x \geqslant 0) \ (*)$$

where a is a constant.

(a) **(i)** Find the solution to (*) in the case $a = 0$ in which $y = 0$ when $x = 0$. **[1]**

(ii) Sketch this solution for $0 \leqslant x \leqslant 5$ in the Printed Answer Booklet. **[1]**

(iii) For this solution, determine the maximum value of y for $0 \leqslant x \leqslant 5$. **[2]**

(b) Fig 4.1 and Fig 4.2 show tangent fields for two distinct but unspecified values of a. In each case a sketch of the solution curve $y = \mathrm{g}(x)$ which passes through the origin is shown for $0 \leqslant x \leqslant 1$.

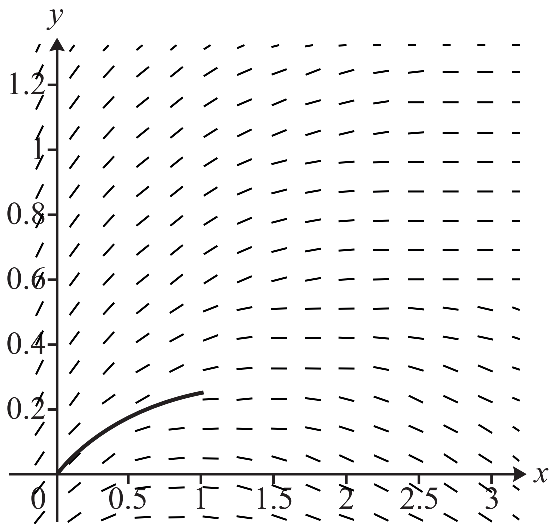

Fig 4.1

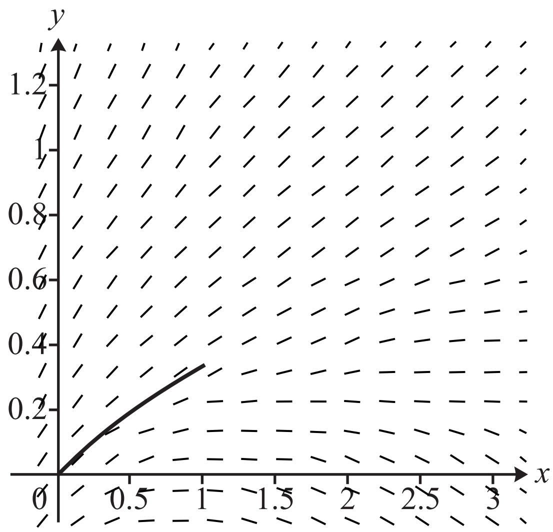

Fig 4.2

(i) For the case in Fig 4.1 suggest a possible value of a. **[1]**

(ii) For the case in Fig 4.2 suggest a possible value of a. **[1]**

(iii) In each case, continue the sketch of the solution curves for $1 \leqslant x \leqslant 3$ in the Printed Answer Booklet. **[2]**

(iv) State a feature which is present in one of the curves in part **(iii)** but not in the other. **[1]**

(c) **(i)** A modified Euler method for the solution of the differential equation $f(x, y) = \frac{dy}{dx}$ is as follows.

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

Construct a spreadsheet to solve (*), so that the value of a and the value of h can be varied, in the case $x_0 = 0$ and $y_0 = 0$. State the formulae you have used in your spreadsheet. **[4]**

(ii) In this part of the question $a = 0.5$. Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

(iii) In this part of the question $a = 1$. Use your spreadsheet with $h = 0.1$ to approximate the value of y when $x = 5$ for the solution to (*) in which $y = 0$ when $x = 0$. **[1]**

There is a value c such that

- if $a > c$ then the solution in which $y = 0$ when $x = 0$ increases without bound as x increases from 0

and

- if $a < c$ then the solution in which $y = 0$ when $x = 0$ increases initially but then peaks and decreases as x increases from 0.

(iv) Use your spreadsheet to find c correct to 2 decimal places. **[4]**

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